CURVATURE EFFECT ON THERMAL BOUNDARY LAYER IN LAMINAR FLOW

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CERTIFICATE

This is to certify that the present work has been carried out under my supervision and the work has not been submitted elsewhere for a degree.

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NOMENCLATURE

x	Curvilinear co-ordinate along boundary surface
y	Co-ordinate normal to the boundary surface
u	Velocity in x - direction in the boundary layer
v	Velocity in y - direction in the boundary layer
K	Surface curvature = $(\text{radius})^{-1}$, (convex positive)
L	Characteristic length
Re	Reynolds number, $U_\infty L/\nu$
U_∞	Positive constant reference velocity
U_1	Tangential velocity at the surface
U	Velocity in x - direction in the potential flow region
Ω	Vorticity
θ	Non-dimensional temperature
T_0	Temperature of surface
T_∞	Temperature of fluid at large distance from surface (reference temperature)
Pr	Prandtl number
ψ	Stream function
A	Curvature parameter
f	Non-dimensional stream function
k	Thermal conductivity of the fluid
p	Pressure

ρ	Density
μ	Dynamic viscosity
ν	Coefficient of kinematic viscosity
α	Thermal diffusivity
η	Non-dimensional distance from wall
$U(x,y)$	Irrotational flow velocity
$U_o(x)$	Irrotational velocity at wall
ω	Angular velocity

ABSTRACT

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Steady heat transfer in laminar flow of a viscous, incompressible fluid over a two dimensional curved surface is investigated, for conditions such that the energy equation has similar solutions. Viscous dissipation is neglected and thermal properties of the fluid are assumed to be constant. Thermal boundary layer equations have been obtained for small and large curvatures from the exact energy equation in curvilinear co-ordinate system. Velocity boundary layer equations obtained by Massey[(2)] for the above flow problem have been resolved by the quasilinearization technique for the stream function and its derivatives. The results thus obtained are used to solve the thermal boundary layer equations numerically using Runge - Kutta method of integration. Results, correct to seven significant figures, are obtained and have been presented graphically for a range of parameters. Comparision of heat transfer rates for concave and convex surfaces are made with flat surface case.

CHAPTER I

INTRODUCTION

The flow of viscous fluid past heated curved surfaces is of great practical importance. In numerous heat transfer devices, a fluid is either heated or cooled while flowing along a surface which is curved in the direction of flow. Some of the familiar examples are: (1) aerodynamic heating of bodies (2) cooling of gas turbine blades (3) flow along airfoil surfaces and (4) flow along rocket nozzles etc. In the design of such an equipment it is important to evaluate the heat transfer coefficient.

Flow of heat is always superimposed on the physical motion of the fluid. The major part of the transition from the temperature of the hot body to that of the cooler surrounding takes place in a thin layer in the neighbourhood of the body which is termed as thermal boundary layer.

In order to determine temperature distribution in the thermal boundary layer region it is necessary to combine the equations of motion with those of heat conduction. It is necessary therefore to account for the work done towards the analysis of equation of motion in a laminar boundary layer on curved surfaces before an attempt could be made for heat transfer analysis.

Murphy[1] has studied velocity boundary layer equations applicable to large and moderate curvature, for laminar flow of an incompressible fluid flowing on a two dimensional surface possessing longitudinal surface curvature proportional to the inverse of the square root of distance from the front stagnation point. The equations of motion were developed in coordinates parallel and normal to the surface with the origin at the front stagnation point. To the equation so developed an order of magnitude analysis was performed and terms of the lowest and one higher order were retained to develop governing equations. Partial differential equations so obtained were converted to ordinary differential equations which were solved by series solution method. Later, Massey and Clayton[2], first differentiated the momentum equations to eliminate the pressure term and then made an order of magnitude analysis, retained highest and next to highest order terms and obtained a single ordinary differential equation derived by the method of "similar solution", and obtained results in agreement with those of Murphy[1]. Further, Massey and Clayton[5] have considered flow over permeable curved surfaces and concluded that for a given curvature, blowing reduces where as suction increases the magnitude of the adverse pressure gradient which boundary layer can with stand before separation occurs. Suction reduces the boundary layer thickness and increases the skin friction; blowing has the reverse effect.

Thus, some work is reported on the effect of longitudinal surface curvature on the skin friction. Also considerable work is available in literature on heat transfer from flat surfaces under various conditions. The study on heat transfer from curved surfaces, however, has received very little attention comparatively.

Frank Kreith[12] has studied both theoretically and experimentally the influence of curvature on heat transfer to incompressible fluids. He has studied heating of fluid in a curved channel of rectangular cross section having circular circurature. Both concave and convex surfaces have been taken to be of the same radius of curvature. It has been concluded that heat transfer rate from concave surface is more than that for the convex surface under similar flow conditions.

Recently, Cheng and Akiyama[13] have studied laminar forced convection heat transfer in curved rectangular channels. In this case the radius of curvature of concave surface is more than that for the convex surface. It has been shown that local heat transfer coefficient is higher at outer wall (concave surface) of the curved channel than at the inner wall (convex surface).

The purpose of the present analysis is to study the effect of curvature on heat transfer for fluids flowing on a heated surface of concave or convex type where the radius of curvature is proportional to the square root of the distance from the stagnation point. The range of curvature that has

been chosen for analysis is the range of practical interest. The method of similar solutions has been used.

Chapter II deals with the formulation of the problem. General energy equation and hence the thermal boundary layer equation have been derived in curvilinear coordinate system. Viscous dissipation is neglected. Velocity boundary layer equation derived by Massey[2] in curvilinear coordinates system has been reviewed.

In Chapter III, the numerical methods of solution of both the velocity boundary layer and the thermal boundary layer equations have been discussed. Chapter IV deals with numerical results which have been tabulated and plotted graphically.

CHAPTER II

FORMULATION OF THE PROBLEM

2.1 Statement of the Problem:

Consider two dimensional flow of an incompressible fluid over a curved surface maintained at a constant temperature. The following assumptions are made:

i) Potential velocity at the surface is constant.*

ii) Physical and thermal properties of the fluid are independent of temperature, so that the momentum and energy equations remain uncoupled.

iii) Temperature of the fluid at large distance from the wall (outside the thermal boundary layer) is constant.

iv) There is no suction or injection.

v) Vorticity is zero outside the boundary layer.

vi) Wall curvature is neither too large nor too small that is it is moderate.

An orthogonal coordinate system is chosen as shown in Figure (1). Distances x and y are measured along and perpendicular to the surface of the body respectively. The front stagnation point is chosen as origin. Local curvature K(x) is assumed positive for flow on convex side and negative for flow on the concave side.

---

* This assumption is justified for moderate curvature surfaces for which $K\delta < 1$, explained later. The radius of curvature is large enough to assume the velocity constant at the surface.
This assumption also makes it possible to compare our results with the Blasius solution for a flat plate case when curvature is zero.

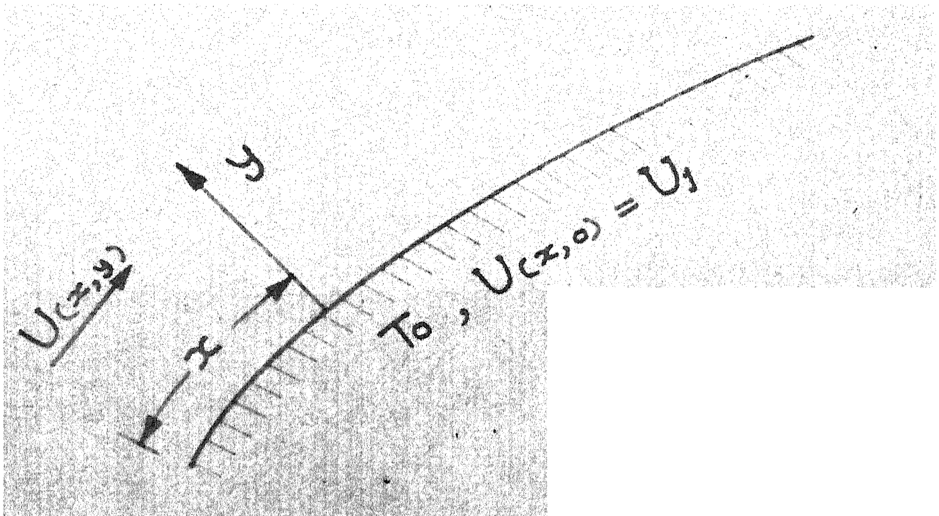

FIGURE-1 COORDINATE SYSTEM

2.2 Derivation of General Energy Equation in Curvilinear Coordinate System:

General energy equation in cartesian coordinates for temperature dependent conductivity is given by:*

$$\rho g C_p \frac{DT}{Dt} = \frac{Dp}{Dt} + \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\frac{\partial T}{\partial z}\right)$$

$$+ \mu \Phi \qquad (2.1)$$

The first term on the right hand side represents the work done due to compression and is equal to zero for incompressible fluids. Neglecting viscous dissipation, the energy equation for steady incompressible flow becomes:

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} + w\frac{\partial T}{\partial z} = \alpha\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right)$$

where the thermal properties are assumed independent of temperature. In vector form, the above equation is:

$$V \cdot \nabla T = \alpha \nabla^2 T \qquad (2.2)$$

The gradient of temperature (∇T) and laplacian ($\nabla^2 T$), can be expressed in terms of curvilinear coordinates (Appendix-A) in two dimensions as:

$$\nabla T = \frac{1}{h_1}\frac{\partial T}{\partial u_1} e_1 + \frac{1}{h_2}\frac{\partial T}{\partial u_2} e_2 \qquad (2.3.1)$$

* Derivation is given on page 291, Boundary Layer Theory by H. Schlichting.

and

$$\nabla^2 T = \frac{1}{h_1 h_2}\left[\frac{\partial}{\partial u_1}\left(\frac{h_2}{h_1}\frac{\partial T}{\partial u_1}\right) + \frac{\partial}{\partial u_2}\left(\frac{h_1}{h_2}\frac{\partial T}{\partial u_2}\right)\right] \qquad (2.3.2)$$

With reference to our coordinate system Figure (2): $u_1 = x$, $u_2 = y$, and e_1 , e_2 are unit tangent vectors in the x and y directions respectively.

Let $PQ = dr$ represent a differential length at a point P and $K = K(x)$ denote the curvature of the surface.

The element of length along the curve parallel to AB , through P is PS and that along the normal to the parallel curve is QS. Draw a line BN parallel to AP. If θ represents the angle subtended by the differential length PQ at the center of curvature, we have:

$$\sin\theta = \theta = \frac{AB}{R} = \frac{NS}{NB} \quad \text{for small } \theta$$

This gives

$$NS = \frac{1}{R} \cdot NB \cdot AB = Ky\,dx$$

$$PS = PN + NS$$

$$= (1+Ky)dx$$

$$QS = dy$$

Therefore:

$$dr = (1+Ky)dx\; e_1 + dy\; e_2$$

also from Appendix-A

$$dr = h_1 du_1 e_1 + h_2 du_2 e_2$$

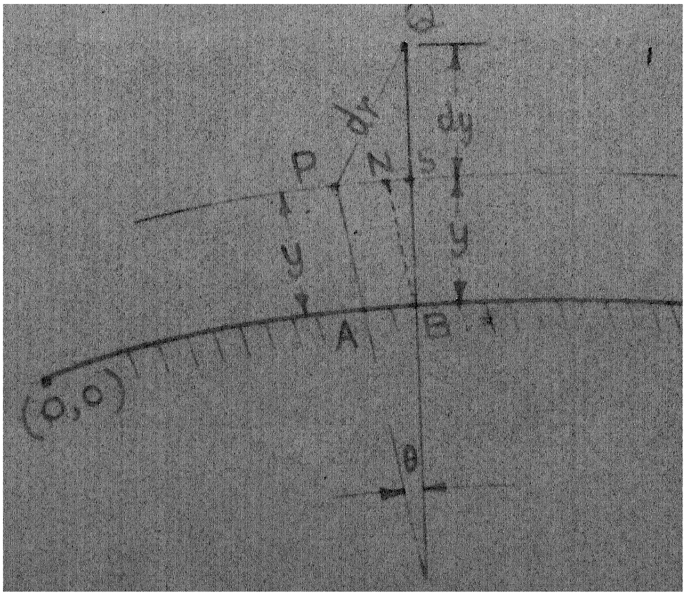

FIGURE - 2

Hence:

$$h_1 = 1+Ky$$

$$h_2 = 1$$

The velocity vector $\bar{V}$ is given by :

$$\bar{V} = ue_1 + ve_2$$

where u and v are velocity components along and perpendicular to the curved wall.

The Gradient and Laplacian of the temperature can now be expressed as:

$$\nabla T = \frac{1}{1+Ky} \frac{\partial T}{\partial x} e_1 + \frac{\partial T}{\partial y} e_2 \qquad (2.4.1)$$

$$\nabla^2 T = \frac{1}{1+Ky} \left[\frac{\partial}{\partial x} \left(\frac{1}{1+Ky} \frac{\partial T}{\partial x} \right) + \frac{\partial}{\partial y} \left((1+Ky) \frac{\partial T}{\partial y} \right) \right]$$

$$= \frac{1}{1+Ky} \left[\frac{\partial^2 T}{\partial x^2} \frac{1}{1+Ky} - \frac{y}{(1+Ky)^2} \frac{dK}{dx} \frac{\partial T}{\partial x} \right.$$

$$\left. + K \frac{\partial T}{\partial y} + (1+Ky) \frac{\partial^2 T}{\partial y^2} \right] \qquad (2.4.2)$$

The energy equation in orthogonal curvilinear coordinate system becomes:

$$(ue_1 + ve_2) . \left(\frac{1}{1+Ky} \frac{\partial T}{\partial x} e_1 + \frac{\partial T}{\partial y} e_2 \right)$$

or

$$= \alpha \nabla^2 T \qquad (2.5.1)$$

$$\frac{u}{1+Ky}\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left[\frac{1}{(1+Ky)^2}\frac{\partial^2 T}{\partial x^2} - \frac{y}{(1+Ky)^3}\frac{dK}{dx}\frac{\partial T}{\partial x} + \frac{K}{1+Ky}\frac{\partial T}{\partial y} + \frac{\partial^2 T}{\partial y^2}\right] \qquad (2.5.2)$$

2.3 Derivation of Thermal Boundary Layer Equation in Curvilinear Coordinates:

Define non-dimensional quantities as follows:

$$u^* = \frac{u}{U_\infty} \quad , \quad x^* = \frac{x}{L}$$

$$v^* = \frac{v}{U_\infty} \quad , \quad K^* = KL$$

$$y^* = \frac{y}{L} \quad , \quad p^* = \frac{p}{\rho U_\infty^2}$$

$$T^* = \frac{T}{T_\infty} \qquad (2.6)$$

Introducing these non-dimensional quantities in equation (2.5.2) we get:

$$\frac{u^* U_\infty T_\infty}{L\left(1+\frac{K^* y^* L}{L}\right)}\frac{\partial T^*}{\partial x^*} + \frac{v^* U_\infty T_\infty}{L}\frac{\partial T^*}{\partial y^*}$$

$$= \alpha\left[\frac{T_\infty}{\left(1+\frac{K^* y^* L}{L}\right)^2}\frac{\partial^2 T^*}{L^2 \partial x^{*2}} + \left\{\frac{\frac{K^* T_\infty}{L}}{\left(1+\frac{K^* y^* L}{L}\right)L}\right\}\frac{\partial T^*}{\partial y^*}\right.$$

$$\left. - \frac{y^* L \,\frac{1}{L}\cdot\frac{1}{L}}{\left(1+\frac{K^* y^* L}{L}\right)^3}\frac{dK^*}{dx^*}\left(\frac{T_\infty}{L}\frac{\partial T^*}{\partial x^*}\right) + \frac{T_\infty}{L^2}\left(\frac{\partial^2 T^*}{\partial y^{*2}}\right)\right] \qquad (2.7)$$

or

$$\frac{u^*}{(1+K^*y^*)}\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*} = \frac{\alpha/\nu}{U_\infty L/\nu}\left[\frac{1}{(1+K^*y^*)^2}\frac{\partial^2 T^*}{\partial x^{*2}}\right.$$

$$\left. + \frac{K^*}{(1+K^*y^*)}\frac{\partial T^*}{\partial y^*} - \frac{y^*}{(1+K^*y^*)^3}\frac{dK^*}{dx^*}\frac{\partial T^*}{\partial x^*} + \frac{\partial^2 T^*}{\partial y^{*2}}\right]$$

or

$$\frac{u^*}{(1+K^*y^*)}\frac{\partial T^*}{\partial x^*} + v^*\frac{\partial T^*}{\partial y^*} = \frac{1}{RePr}\left[\frac{1}{(1+K^*y^*)^2}\frac{\partial^2 T^*}{\partial x^{*2}} + \frac{K^*}{(1+K^*y^*)}\frac{\partial T^*}{\partial y^*}\right.$$

$$\left. - \frac{y^*}{(1+K^*y^*)^3}\frac{dK^*}{dx^*}\frac{\partial T^*}{\partial x^*} + \frac{\partial^2 T^*}{\partial y^{*2}}\right] \quad (2.8)$$

The order of magnitude of various parameters in the above equation is indicated as:

$$u^* \sim O(1) \qquad v^* \sim O(\delta)$$

$$\frac{\partial u^*}{\partial x^*} \sim O(1) \qquad \frac{\partial v^*}{\partial y^*} \sim O(1)$$

$$\frac{\partial^2 u}{\partial x^{*2}} \sim O(1) \qquad \frac{\partial v^*}{\partial x^*} \sim O(\delta) \quad (2.9)$$

$$\frac{\partial p^*}{\partial x^*} \sim O(1) \qquad \frac{\partial^2 v^*}{\partial x^*\partial y^*} \sim O(1)$$

$$\frac{\partial p^*}{\partial x^*} \sim O(1)$$

For moderate curvature $\quad K^* \sim O(1)$

For large curvature $\quad K^* \sim O(\frac{1}{\delta})$

For moderate curvature $\qquad \frac{\partial T^*}{\partial x^*} \sim O(1)$

For large curvature $\qquad \frac{\partial T^*}{\partial x^*} \sim O(1)$

For moderate curvature, the order of magnitude analysis of various terms in equation (2.8) is:

$$\frac{1}{(1+1.\ )} + \delta \frac{1}{\delta} = \frac{1}{RePr}\left[\frac{1}{(1+1.\delta)^2} + \frac{1}{(1+\delta)}\frac{1}{\delta} + \frac{1}{\delta^2} - \frac{\delta}{(1+\delta)^3}\right]$$

or

$$O(1) + O(1) = \frac{1}{RePr}\left[O(1) + O(\tfrac{1}{\delta}) - O(\delta) + \frac{1}{O(\delta^2)}\right] \qquad (2.10)$$

Retaining the highest order terms on the right hand side of equation (2.10), the thermal boundary layer equation in the dimensional form for moderate curvature becomes:

$$\frac{u}{(1+Ky)}\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (2.11)$$

When $K = 0$, this reduces to thermal boundary layer equation for the flat plate case.

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\frac{\partial^2 T}{\partial y^2} \qquad (2.12)$$

Similarly, for large curvature, the order of magnitude analysis of various terms in equation (2.8) is:

$$O(1) + O(1) = \frac{1}{RePr}\left[O(1) + \frac{1}{O(\delta^2)} - O(1) + \frac{O(1)}{O(\delta^2)}\right]$$

Retaining highest order terms on the right hand side, the thermal boundary layer equation for large curvature becomes:

$$\frac{u}{1+Ky}\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y} = \alpha\left[\frac{\partial^2 T}{\partial y^2} + \frac{K}{1+Ky}\frac{\partial T}{\partial y}\right] \tag{2.13}$$

This also reduces to equation (2.12) when $K = 0$.

2.4 Velocity Boundary Layer Equation in Curvilinear Coordinates:

The governing equations of motion for two dimensional flow in curvilinear coordinate system are:

Continuity Equation:

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial y}\left[(1+Ky)\,v\right] = 0 \tag{2.14}$$

Momentum equation in the x - direction:

$$\begin{aligned}\frac{1}{1+Ky}u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \frac{Kuv}{1+Ky} = &-\frac{1}{\rho(1+Ky)}\frac{\partial p}{\partial x}\\ &+\nu\left[\frac{1}{(1+Ky)^2}\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{y}{(1+Ky)^3}\frac{dK}{dx}\frac{\partial u}{\partial x}\right.\\ &+\frac{K}{1+Ky}\frac{\partial u}{\partial y} - \frac{K^2 u}{(1+Ky)^2} + \frac{2K}{(1+Ky)^2}\frac{\partial v}{\partial x}\\ &\left.+\frac{v}{(1+Ky)^3}\frac{dK}{dx}\right]\end{aligned} \tag{2.15}$$

Momentum equation in the y - direction:

$$\frac{1}{1+Ky} u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - \frac{Ku^2}{1+Ky} = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{1}{(1+Ky)^2}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{y}{(1+Ky)^3}\frac{dK}{dx}\frac{\partial v}{\partial x} + \frac{K}{1+Ky}\frac{\partial v}{\partial y} - \frac{K^2 v}{1+Ky} - \frac{2K}{(1+Ky)^2}\frac{\partial u}{\partial x} - \frac{u}{(1+Ky)^3}\frac{dK}{dx}\right], \quad (1+Ky \neq 0) \qquad (2.16)$$

Define a stream function $\psi(x,y)$ such that

$$u = \frac{\partial \psi}{\partial y}$$

$$v = -\frac{1}{1+Ky}\frac{\partial \psi}{\partial x}$$

This satisfies the continuity equation (2.14).

Differentiating equations (2.15) and (2.16) with respect to y and x respectively and eliminating pressure term and then performing order of magnitude analysis using non-dimensional quantities defined in section (2.3), Massey[2] obtained a single equation for the velocity boundary layer for moderate curvatures.

$$\psi_{yyx}\psi_y - \psi_x\psi_{yyy} - \frac{K}{1+Ky}\psi_x\psi_{yy} + \frac{K}{1+Ky}\psi_y\psi_{xy} + \frac{K_x}{(1+Ky)^2}\psi_y\psi_y + \frac{K^2}{(1+Ky)^2}\psi_y\psi_x = \nu\left[(1+Ky)\psi_{yyyy} + 2K\psi_{yyy} - \frac{K^2}{1+Ky}\psi_{yy} + \frac{K^3}{(1+Ky)^2}\psi_y\right] \qquad (2.17)$$

CHAPTER III

BOUNDARY - LAYER SOLUTIONS

3.1 Velocity Boundary Layer Equation:

The velocity boundary layer and the thermal boundary layer equations are difficult to solve in their original forms given by equations (2.17) and (2.11) respectively. In order to simplify the analysis, we introduce a similarity variable and proceed in the following manner:

Consider an idealised flow past the surface, such that the potential flow velocity at the surface is constant and is equal to U_1. Also consider a family of surfaces having curvature distribution given by:

$$K = A\sqrt{\frac{U_1}{\nu x}} \tag{3.1}$$

where A is a curvature parameter. To replace the partial differential equation by an ordinary differential equation, define a new dimensionless variable called similarity variable as:

$$\eta = y\sqrt{\frac{U_1}{\nu x}} \tag{3.2}$$

Let a stream function be defined as:

$$\psi(x,y) = \sqrt{U_1 \nu x}\;\; f(\eta) \tag{3.3}$$

where, f is called dimensionless stream function and is the function of η only.

Differentiating equation (3.1) with respect to x, we get

$$\frac{dK}{dx} = -\frac{A}{2x}\sqrt{\frac{U_1}{\nu x}} \tag{3.4}$$

Differentiating the stream function $\psi(x,y)$ with respect to x, y as many times as shown by subscripts on ψ we get the following:

$$\psi_y = \sqrt{U_1 \nu x}\ \frac{df}{d\eta}\frac{\partial\eta}{\partial y}$$

$$= \sqrt{U_1 \nu x}\sqrt{\frac{U_1}{\nu x}}\ f'$$

$$= U_1 f' \tag{3.5}$$

$$\psi_{yy} = U_1\sqrt{\frac{U_1}{\nu x}}\ f'' \tag{3.6}$$

$$\psi_{yyy} = U_1\left(\frac{U_1}{\nu x}\right) f''' \tag{3.7}$$

$$\psi_{yyyy} = U_1\left(\frac{U_1}{\nu x}\right)^{3/2} f'''' \tag{3.8}$$

$$\psi_{yx} = U_1\ f'' y\sqrt{\frac{U_1}{\nu}}\left(-\frac{1}{2x^{3/2}}\right)$$

$$= -\frac{U_1\eta}{2x} f'' \tag{3.9}$$

$$\psi_{yyx} = -\frac{U_1}{2x}\sqrt{\frac{U_1}{\nu x}}\left(f'' + \eta f'''\right) \tag{3.10}$$

$$\psi_x = -\frac{1}{2}\sqrt{\frac{U_1\nu}{x}}\ (f'_\eta - f) \tag{3.11}$$

Substituting into the velocity boundary layer equation (2.17), we get for moderate curvatures,

$$
\begin{aligned}
f'''' &+ \frac{2A}{(1+A\eta)} f''' - \frac{A^2 f''}{(1+A\eta)^2} + \frac{A^3 f'}{(1+A\eta)^3} \\
&+ \frac{f'f''}{2(1+A\eta)} + \frac{f\,f''}{2(1+A\eta)} + \frac{A}{2}\frac{f\,f''}{(1+A\eta)^2} \\
&+ \frac{Af'f'}{2(1+A\eta)^3} - \frac{A^2 f'(f-\eta f')}{2(1+A\eta)^3} = 0 \qquad (3.12)
\end{aligned}
$$

For $A = 0$, the above equation reduces to

$$2f'''' + f'f'' + f\,f''' = 0 \qquad (3.13)$$

Integrating (3.13) we get :

$$2f''' + f\,f'' = 0 \qquad (3.14)$$

This is well known Blasius equation for flat plate.

3.2 <u>Boundary Conditions for Equation</u> (3.12):

Vorticity, Ω in two dimensional flow for a curved surface is given by (Appendix-B):

$$\Omega = \frac{1}{1+Ky}\left[\frac{\partial V}{\partial x} - \frac{\partial}{\partial y}\left\{(1+Ky)\,U\right\}\right] \qquad (3.15)$$

where U is the velocity distribution in potential flow. If the flow outside the boundary layer is assumed to be potential, the vorticity should be zero. This gives (from equation (3.15))

$$\frac{\partial V}{\partial x} - \frac{\partial}{\partial y}\left\{(1+Ky)\,U\right\} = 0 \qquad (3.16)$$

It can be easily seen that the term $\frac{\partial V}{\partial x}$ is very small in external potential flow, and hence can be neglected as compared to the other term.

Hence:

$$\frac{\partial}{\partial y}\left[(1+Ky)\,U\right] = 0 \qquad (3.17)$$

Integration of equation (3.17) gives:

$$(1+Ky)\,U = \text{constant or function of } x \qquad (3.17a)$$

In general, we may assume that the potential flow velocity at the wall ($y = 0$) is a function of x say $Uo(x)$.

For evaluating the constant of equation (3.17a), the above condition gives:

$$(1+Ky)\,U = Uo(x)$$

that is:

$$U = \frac{Uo(x)}{1+Ky} \qquad (3.18)$$

However, we have already assumed an idealised flow past the surface such that the potential velocity at the surface is constant $= U_1$

Hence:

$$U = \frac{U_1}{1+Ky} \qquad (3.18a)$$

Differentiating partially with respect to y, we get:

$$\frac{\partial U}{\partial y} = -\frac{U_1 K}{(1+Ky)^2} \qquad (3.19)$$

This gives rate of change of potential velocity in the x - direction with respect to y and must be approximately equal to $\frac{\partial u}{\partial y}$ at the edge of the boundary layer, where u is the velocity distribution in the boundary layer region.

Hence:

$$\frac{\partial u}{\partial y} = -\frac{U_1 K}{(1+Ky)^2} = \psi_{yy} \qquad (3.20)$$

Using (3.6), we get:

$$U_1 \sqrt{\frac{U_1}{\nu x}}\, f'' = -\frac{U_1 K}{(1+Ky)^2} \qquad (3.21)$$

Substituting for K and y from equations (3.1) and (3.2) in (3.21) and simplifying:

$$f'' = -\frac{A}{(1+A\eta)^2} \qquad (3.22)$$

Also, at the edge of the boundary layer the velocity component u in the x - direction should be equal to the potential velocity distribution U at any point of the edge of the boundary layer. Thus $u = U(x,y)$ at the edge of the boundary layer.

From equation (3.5)

$$u = \psi_y = U = U_1\ f' \qquad (3.23)$$

Equating (3.18a) and (3.23), we get:

$$f' = \frac{1}{1+Ky} \qquad (3.24)$$

Using expressions (3.1) and (3.2), the relation (3.24) is transformed into

$$f' = \frac{1}{1+A\eta} \tag{3.24a}$$

We also have the velocity component in y - direction in the boundary layer region as:

$$v = -\frac{1}{1+Ky}\frac{\partial \psi}{\partial x}$$

$$= -\frac{1}{1+A\eta}\sqrt{\frac{U_1 \nu}{4x}}\left(f-\eta f'\right) \tag{3.25}$$

No slip condition at the wall is:

$$\left.\begin{aligned} u &= 0 \\ v &= 0 \end{aligned}\right\} \quad \text{at } \eta = 0$$

Using these conditions, the equations (3.5) and (3.25) give:

$f' = 0$ at the wall ($\eta = 0$)

$f = 0$ at the wall ($\eta = 0$)

Hence, boundary conditions for the equation (3.12) are:

$$\begin{aligned} f &= 0, \quad &\text{at } \eta = 0 \\ f' &= 0 \quad &\text{at } \eta = 0 \end{aligned} \tag{3.26}$$

and

$$\left.\begin{aligned} f' &= \frac{1}{1+A\eta}, \\ f'' &= -\frac{A}{(1+A\eta)^2} \end{aligned}\right\} \quad \text{at } \eta = \eta_e \tag{3.27}$$

where η_e is the value of η at the edge of the boundary layer.

At the point of separation one more additional boundary condition is prescribed that is:

$$f'' = 0 \quad \text{at } \eta = 0 \tag{3.28}$$

Thus the velocity boundary layer equation (3.12) can be solved for moderate curvatures with boundary conditions given in equations (3.26), (3.27) and (3.28). The velocity distributions in the boundary layer and at the point of separation have been found in section (3.4).

3.3 Thermal Boundary Layer Equation:

The thermal boundary layer equation (2.11) for moderate curvature may similarly be expressed as an ordinary differential equation by using the same variables as defined in equations (3.1), (3.2) and (3.3) and a non-dimensional temperature:

$$\theta = \frac{T_0 - T}{T_0 - T_\infty} \tag{3.29}$$

Equation (3.29) gives :

$\theta = 0$, at the wall

$\theta = 1$, at the edge of the boundary layer

Differentiating equation (3.29) with respect to x and using equation (3.2), we get:

$$\frac{\partial T}{\partial x} = -(T_0 - T_\infty)\frac{\partial \theta}{\partial x} = -(T_0 - T_\infty)\,\theta'\frac{\partial \eta}{\partial x}$$

$$= (T_0 - T_\infty)\frac{\eta}{2x}\,\theta' \tag{3.30}$$

Also,

$$\frac{\partial T}{\partial y} = -(T_0 - T_\infty)\,\theta'\frac{\partial \eta}{\partial y}$$

Using expression (3.2):

$$\frac{\partial T}{\partial y} = -(T_0 - T_\infty)\sqrt{\frac{U_1}{\nu x}}\ \theta' \qquad (3.31)$$

and

$$\frac{\partial^2 T}{\partial y^2} = -(T_0 - T_\infty)\ \frac{U_1}{\nu x}\ \theta'' \qquad (3.32)$$

where $\theta' = \frac{d\theta}{d\eta}$; $\theta'' = \frac{d^2\theta}{d\eta^2}$

Substituting in the boundary layer equation (2.11), we get:

$$\frac{f'U_1}{1+A\eta}\frac{\eta}{2x}\,\theta' + \left(-\frac{1}{1+A\eta}\sqrt{\frac{U_1\nu}{4x}}\right)(f-\eta f')\left(-\sqrt{\frac{U_1}{\nu x}}\ \theta'\right)$$

$$= -\alpha\left(\frac{U_1}{\nu x}\right)\ \theta''$$

or $\frac{f'\eta\theta'}{2(1+A\eta)} + \frac{1}{2(1+A\eta)}(f-\eta f')\ \theta' = -\frac{\alpha}{\nu}\ \theta''$

or

$$\frac{f\theta'}{2(1+A\eta)} + \frac{\alpha}{\nu}\ \theta'' = 0$$

or

$$\theta'' + \frac{Pr.f.\theta'}{2(1+A\eta)} = 0 \qquad (3.33)$$

For large curvature the thermal boundary layer equation (2.13) is used and using the expressions (3.29) to (3.32) we similarly get:

$$\frac{1}{2}\frac{f'\eta\,\theta'}{1+A\eta} + \frac{1}{2(1+A\eta)}\ (f-f'\eta)\quad \theta' = \frac{\alpha}{\nu}\left[-\theta'' - \frac{\theta' A}{1+A\eta}\right]$$

or $$\frac{f\,\theta'}{2(1+A\eta)} = \frac{Pr}{Pr\,Pr}\left(-\theta'' - \frac{\theta' A}{1+A\eta}\right)$$

or $$\theta'' + \frac{\theta'}{1+A\eta}\left(\frac{f\,Pr}{2} + A\right) = 0 \qquad (3.34)$$

Boundary conditions for both the equations (3.33) and (3.34) are :

$$\begin{aligned} &\theta = 0 \quad \text{at} \quad \eta = 0 \quad \text{(at the surface)} \\ &\theta = 1 \quad \text{at} \quad \eta = \eta_e \quad \text{(at the edge of the thermal boundary layer)} \end{aligned} \qquad (3.35)$$

Since our interest lies only in moderate curvatures with the practical view point, we will concentrate our analysis for velocity and thermal boundary layers for moderate curvatures only. The thermal boundary layer equation (3.33) for moderate curvature with boundary conditions (3.35) is therefore to be solved. The values of the non-dimensional stream function f at various values of η are needed in order to seek a solution for this equation.

Although the velocity boundary layer equation (3.12) under the boundary conditions specified in (3.26) has already been solved for (f, f', f'') by previous workers (1,2), we have used a more sophisticated method for its solution as described in the subsequent section.

3.4 <u>Numerical Method of Solution</u>:

(A) Velocity Boundary Layer Equation:

Equation (3.12) is a fourth order non-linear ordinary differential equation. It is very difficult to

obtain exact solution for this equation. It is therefore necessary to seek for the numerical solution. Four boundary conditions are prescribed. Two of the boundary conditions are prescribed at $\eta = 0$ which are called initial conditions and two at large value of η that is at the edge of the boundary layer. Murphy[1] has used crude series method to solve the equation. Later on he[4] himself solved by trial and error scheme using Moulton's method for numerical solution of ordinary differential equation. Massey[2] used trial and error method similar to Murphy[4]. For a fixed value of parameter he selected $f''(0)$, $f'''(0)$ with the help of previous knowledge to start the integration of the equation. At the outer limit, computed values of f', f'' were compared with boundary conditions. Based on the resulting errors at the outer limit, new choices of $f''(0)$ and $f'''(0)$ were made and integration repeated. The iteration process was continued until the accuracy of f' at the outer boundary was better than $\pm 2 \times 10^{-5}$ and that of corresponding f'' better than $\pm 2 \times 10^{-4}$. Runge - Kutta step by step method of integration was used.

In all the above cases, it was necessary to start the integration by selecting values of unknown initial conditions, based on physical intution or otherwise. To find improved boundary conditions, additional solution was obtained with another set of guessed initial conditions. By linear interpolation or extrapolation improved boundary conditions were obtained[10].

The disadvantage of this process is that we never know the most appropriate starting values. The result is that the process may diverge or converge extremely slowly. In addition, certain other problems may arise for example instability of solutions at large values of independent variable. Also small error in the boundary values may result appreciable error in the solution.

In order to avoid the discrepancies inherent in the above methods, equation (3.12) has been solved using quasilinearization technique which has been explained in appendix-C.

The non-linear equation (3.12) is of the form:

$$f'''' + F(\eta, f, f', f'', f''') = 0 \qquad (3.36)$$

and can be written in the quasilinearised form as shown in Appendix-C as follows:

$$f_{n+1}'''' + \left(\frac{\partial F}{\partial f'''}\right)_n f_{n+1}''' + \left(\frac{\partial F}{\partial f''}\right)_n f_{n+1}'' + \left(\frac{\partial F}{\partial f'}\right)_n f_{n+1}'$$

$$+ \left(\frac{\partial F}{\partial f}\right)_n f_{n+1} = - F(\eta, f, f', f'', f''')_n$$

$$+ f_n \left(\frac{\partial F}{\partial f}\right)_n + f_n' \left(\frac{\partial F}{\partial f'}\right)_n$$

$$+ f_n'' \left(\frac{\partial F}{\partial f''}\right)_n + f_n''' \left(\frac{\partial F}{\partial f'''}\right)_n$$

$$- (3.37)$$

The right hand side of the above equation represents the non-homogeneous part. Suffix n represents n^{th} iteration.

The boundary conditions for the equation (3.37) are:

$$f(0) = 0$$

$$f'(0) = 0$$

$f'(\eta_e) = \frac{1}{1+A\eta_e}$, at the edge of the velocity boundary layer

$f''(\eta_e) = -\frac{A}{(1+A\eta_e)^2}$, at the edge of the velocity boundary layer.

The two conditions f'' and f''' are not known at $\eta = 0$. Hence there will be two complementary solutions and one particular integral as explained in Appendix-C.

The initial conditions that is the conditions at $\eta = 0$ for the first complimentary solution Z1 of equation (3.37) with non-homogeneous terms equal to zero are to be taken as:

$$\begin{aligned} f\ (0) &= 0 \\ f'\ (0) &= 0 \\ f''\ (0) &= 1 \\ f'''(0) &= 0 \end{aligned} \qquad (3.38)$$

The initial conditions for second complementary solution Z2 of equation (3.37) with non-homogeneous terms equal to zero are to be taken as:

$$\begin{aligned} f\ (0) &= 0 \\ f'\ (0) &= 0 \\ f''\ (0) &= 0 \\ f'''(0) &= 1 \end{aligned} \qquad (3.39)$$

The initial conditions for the particular solution Z3 of equation (3.37) with non-homogeneous terms not equal to zero are to be taken as:

$$\begin{aligned} f\ (o) &= 0 \\ f'\ (o) &= 0 \\ f''\ (o) &= 0 \\ f'''(o) &= 0 \end{aligned} \qquad (3.40)$$

In all the above three solutions the variable η ranges from $\eta = 0$ at the surface of body to $\eta = 7.1$ at the edge of the boundary layer. The value of η at the edge of the boundary layer may be any value which is significantly large. It has been found that $\eta = 7.1$ is the most suitable choice, beyond which if we choose any value there is no appreciable change in the solution.

The complete solution of equation (3.37) with boundary conditions (3.26) and (3.27) is the linear combination of Z1, Z2 and Z3, and is given by:

$$f = C_1 x Z1 + C_2 x Z2 + Z3 \qquad (3.41)$$

In order to start for obtaining the solution, it has been assumed that the first solution (f, f', f'' f''') itself is zero that is the values of f, f', f'' and f''' are zero at all the station points ($\eta = 0$, 0.1, 0.2, 0.3, . . . to $\eta = 7.1$). Using the above assumed solution ($f = 0$, $f' = 0$, $f'' = 0$, $f''' = 0$), the three solutions Z1, Z2, Z3 are obtained with initial conditions (3.38), (3.39) and (3.40) respectively. Runge-Kutta step by step method has been used for initial value integrations.

In order to get complete solution expressed by equation (3.41), the values of C_1 and C_2 are needed. The constants C_1 and C_2 have been evaluated using outer boundary conditions (3.27) at $\eta = 7.1$ as follows:

$$f'(7.1) = \frac{1}{1 + 7.1 \times A} \tag{3.42}$$

$$f''(7.1) = -\frac{A}{(1 + 7.1 \times A)^2} \tag{3.43}$$

From equations (3.42) and (3.43) :

$$f'(7.1) = C_1 \times Z1' + C_2 \times Z2' + Z3' \tag{3.44}$$

$$f''(7.1) = C_1 \times Z1'' + C_2 \times Z2'' + Z3'' \tag{3.45}$$

Solving equations (3.44) and (3.45), we get the values of C_1 and C_2.

Using the values of C_1 and C_2 thus obtained and the solutions Z1, Z2, Z3 along with their derivatives Z1', Z2', Z3', Z1'', Z2'', Z3'', Z1''', Z2''', Z3''', new values of f, f', f'', f''' are calculated at different stations ($\eta = 0.1, 0.2 \ldots$ to $\eta = 7.0$).

To ascertain whether the solution (f, f', f'', f''') obtained above by linear combination of complimentary and particular solutions, is the correct one, the numerical values of the new solution are compared with the numerical values of the assumed solution at each station point ($\eta = 0.1, 0.2, 0.3 \ldots$ to $\eta = 7.0$). If the difference between the assumed solution and the new solution at any station point is more

than $\pm 10^{-7}$ (say), the assumed solution is replaced by the new solution. The process is repeated with new solution as assumed solution till desired accuracy is achieved. The final solution (f, f', f'', f''') so obtained represents the desired solution of equation (3.12) with boundary conditions (3.26) and (3.27).

(B) Thermal Boundary Layer Equation:

The thermal boundary layer equation (3.33) is a linear homogeneous equation of second order with boundary conditions (3.35). The usual method of solving the two point linear boundary value problem is used. The procedure runs as follows:

Equation (3.33) is rewritten as

$$\theta'' + \frac{Pr\, f\, \theta'}{2(1+A\ )} = 0 \tag{3.33}$$

The initial and boundary conditions are:

$$\left.\begin{aligned} \theta &= 0 \quad \text{at } \eta = 0 \quad \text{(at the wall)} \\ \theta &= 1 \quad \text{at } \eta > 0 \quad \text{(at the edge of the boundary layer)} \end{aligned}\right\} \tag{3.35}$$

As obvious, only one initial condition on θ is known. Hence there will be only one complementary solution and one particular solution as explained in Appendix-C. Since, equation (3.33) is a second order equation, we need two initial conditions to obtain either the complimentary or the particular solution.

The initial conditions for the complementary solution ZZ1, of equation (3.33) are to be taken as:

$$\begin{aligned} \theta \quad (o) &= 0 \text{ , given} \\ \theta' \quad (o) &= 1 \text{ , assumed} \end{aligned} \qquad (3.46)$$

and for the particular solution ZZ2 as:

$$\begin{aligned} \theta \quad (o) &= 0 \text{ , given} \\ \theta' \quad (o) &= 0 \text{ , assumed} \end{aligned} \qquad (3.47)$$

In both the solutions (ZZ1, ZZ2), the variable η ranges from $\eta = 0$ at the surface to $\eta = 7.1$ at the edge of the boundary layer, the reason for which has already been stated.

The complete solution of equation (3.33) is the linear combination of ZZ1 and ZZ2 and is given by:

$$\theta = D \times ZZ1 + ZZ2 \qquad (3.48)$$

where D is a constant which is to be determined using the outer boundary condition ($\theta = 1$ at $\eta > 0$).

In order to start for obtaining the solution, the first solution (θ, θ') itself is assumed to be zero that is ($\theta = 0$, $\theta' = 0$) at all the stations ($\eta = 0$, 1, 2 . . . . to $\eta = 7.1$).

Using the above assumed solution ($\theta = 0$, $\theta' = 0$), the two solutions ZZ1, ZZ2 are obtained with initial conditions (3.46) and (3.47) respectively. Runge - Kutta step by step method has been used for initial value integrations. The constant D is evaluated for the outer boundary condition at $\eta = 7.1$.

$$\theta(7.1) = D \times ZZ1(7.1) + ZZ2(7.1) \qquad (3.49)$$

This gives the value of the constant D. Using the values of D, Z21, Z22 a new solution (θ, θ') is obtained.

Numerical values of this new solution are compared with those of the assumed solution at each station point. The process is repeated with new solution as assumed solution till desired accuracy is achieved.

It is to be noted that the non-dimensional stream function "f" in equation (3.33) is taken from the solution of the velocity boundary layer equation (3.12) and the thermal boundary layer equation is solved for (θ, θ') for a range of Prandtl numbers.

3.5 Velocity and Temperature Distribution at Separation:

At the point of separation, the shear stress is zero that is $f'' = 0$. This provides an additional initial condition

$$f''(0) = 0$$

The quasilinearised velocity boundary layer equation (3.37) is now solved in the manner already described under the following initial and boundary conditions:

$$f(0) = 0$$

$$f'(0) = 0$$

$$f''(0) = 0$$

$$f' = \frac{1}{1+A\eta} \text{ at the edge of the boundary layer}$$

The solution will comprise of a complementary solution with initial conditions

$$f\ (0) = 0$$
$$f'\ (0) = 0$$
$$f''\ (0) = 0$$
$$f'''\ (0) = 1$$

and a particular solution with initial conditions

$$f\ (0) = 0$$
$$f'\ (0) = 0$$
$$f''\ (0) = 0$$
$$f'''\ (0) = 0$$

The final solution for the stream function f, thus obtained at separation has been used to find temperature distribution at separation for the thermal boundary layer equation (3.33) under the conditions (3.35). The same numerical technique is repeated in the above mentioned integrations.

CHAPTER IV

RESULTS AND DISCUSSION

Before proceeding for discussion on actual velocity and temperature distribution, for curved surfaces, it is desirable to testify the accuracy of the numerical technique we have followed. This has been done for the special case of a flat plate for which the curvature $A = 0$. The velocity distribution results have been tabulated in Table (I) and plotted against η in Figure (5) for various iterations as explained in section 3.4. It is seen that the seventh iteration gives the best results which compares very well with the available Blasius solution for the velocity profile for a flat plate. Also, as seen from Figure (10), the velocity gradient f'' at the wall obtained by this method reads 0.333 as against $f''(0) = 0.332$ obtained by Blasius.

<u>Tangential Velocity Profile</u>:

Velocity distribution has been calculated for a range of curvature parameter ($A = -0.06$ to $+0.06$) for various values of η ($\eta = 0$ to $\eta = 7.1$). The results are tabulated in Table (II) and plotted in Figure (6).

Potential velocity distribution for each curvature has also been shown emerging from the point ($u/U_1 = 1$, $\eta = 0$)*. Equation (3.24a) has been used to plot the potential velocity distribution. The line PQ joins all the points where the potential velocity distribution lines meet the velocity distribution lines in the boundary layer region for different curvatures. This line represents the limit of the boundary layer thickness for each curved surface. It is seen that the boundary layer thickness increases as curvature increases from negative to positive value a result in conformity with that obtained by previous workers[2].

Table (II-A) indicates velocity distribution at separation calculated for various curvatures at different η. The results are plotted in Figure (7). The velocity is zero at the surface and approaches the potential velocity at the edge of the boundary layer as is the case for the velocity distribution in the boundary layer region shown in Figure (6).

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* The range of curvature parameter A has been chosen between $A = -0.06$ to $A = +0.06$ because it lies in the range of practical interest. For example, Figure (6) shows that for $A = +0.06$, the value of η at the outer edge of the boundary layer is about 6.5. Hence $A\eta = 0.39 = K\delta = \frac{\delta}{R}$; that is, the thickness of the boundary layer is about 0.4 times the radius of curvature R. Similarly, for $A = 0.02$, η at the outer edge of the boundary layer is about 5.6. This gives $\delta = 0.112$ R. However, when $A > 0.06$, it can be seen that the thickness of the boundary layer is of the order of the radius of curvature which is not desirable. Hence the range ($A = -0.06$ to $A = +0.06$) seems to be a range of practical interest.

Also the gradient at $\eta = 0$ is zero in this case as expected. This physically means that the shear stress at the point of separation is zero. It can be compared easily from Figures (6) and (7), that the magnitude of velocity at any η for a particular curvature is more before separation as it should be.

It is to be noted from Figures (6) and (7), that the effect of curvature on velocity distribution is more pronounced at values of $\eta > 2$ that is away from the surface. Also the profile shape gets modified for large η due to the outer boundary condition.

In Figure (8), the effect of curvature is plotted on the tangential velocity gradient f'' at the surface ($\eta = 0$). This shows that the shear stress for convex surfaces is less than the flat plate value while that for concave surfaces is greater than the flat plate value. It means that the convex surface is more sensitive to an adverse pressure gradient $\left[(\partial p/\partial x) > 0 \right]$ than the concave surface. In otherwords, for a convex surface, the shear stress being less, the separation will take place earlier if there is an adverse pressure gradient.

__Temperature Profile:__

Temperature distribution has been evaluated for various values of curvature parameter ($A = -0.06$, $A = 0$ and $A = +0.06$) at different η . The results have been tabulated in Tables (III, III-A, IV and IV-A) for Prandtl

numbers 0.1, 1.0 and 5.0, and have been plotted in Figures (9) and (10). It is seen that for a fixed η, the temperature is more before separation. The effect of curvature, however, is of the same nature before and at separation; that is, for the same value of η, the temperature is more for concave surface than for the convex surface. The effect of variation of Prandtl number is also similar before and at separation. The temperature in both the cases increases with an increase in the Prandtl number. As expected, the temperature of the free stream is attained quicker at higher Prandtl numbers and hence the boundary layer thickness is also smaller than at low Prandtl numbers, irrespective of the surface being concave or convex. Comparing Figures (9) and (10) it can be noted that the boundary layer thickness is greater at separation than before separation - a result similar to the velocity boundary layer.

Tables (V) and (V-A) furnish results for the temperature gradient at the wall, $\theta'(0)$ at various Prandtl numbers for various curvatures before separation and at separation. The results have been plotted in Figure (11). It is seen that the temperature gradient at the wall and hence the heat transfer rate is increasing with the increase of Prandtl number both for convex and concave surfaces before and at separation. Also, as compared to the flat plate case, the heat transfer rate is more for the concave surface than for the concex surface. This is expected because the thermal

boundary layer thickness for concave surface is less as compared to that for the convex surface.

CONCLUSION :

The following conclusions can be drawn from the results obtained in the present analysis :

1. The quasilinearization technique of solving the velocity boundary layer equation is a better and more sophisticated a method as compared to trial and error scheme used by previous workers. The present method gives the result quicker and is less cumbrous because there is no need of guessing the initial condition each time. Also the boundary condition at the outer edge of the boundary layer is fully satisfied in the present method. The results obtained are in very good agreement with those of Massey[2].
2. Effect of curvature on heat transfer has been investigated for surfaces with radius of curvature proportional to square root of distance from the stagnation point, using the method of similar solutions. The heat transfer rate is more for a concave surface than for the convex surface. The result is similar in nature that obtained by previous workers for some particular situations.

A similar analysis may be made for heat transfer from curved surfaces with suction or injection.

TABLE I

VELOCITY DISTRIBUTION AT VARIOUS ITERATIONS FOR A=0.0

η \ ITERATION	FIRST	SECOND	THIRD	FIFTH	SEVENTH
0.00	0.00000000	0.00000000	0.00000000	0.00000000	0.00000000
0.10	0.02797066	0.03253548	0.03333578	0.03336236	0.03336236
0.50	0.13588584	0.16299600	0.16649991	0.16662766	0.16662768
1.00	0.26185301	0.32521990	0.33090390	0.33114573	0.33114579
1.50	0.37790149	0.48202377	0.48828611	0.48861996	0.48862005
2.00	0.48403129	0.62659610	0.63146961	0.63185723	0.63185790
2.50	0.58024239	0.75156607	0.75299653	0.75338276	0.75338295
3.00	0.66653480	0.85109712	0.84766653	0.84799007	0.84799026
3.50	0.74290855	0.92272956	0.91445445	0.91466811	0.91466828
4.00	0.80936363	0.96813570	0.95668869	0.95677871	0.95677882
4.50	0.86589996	0.99240832	0.98044336	0.98043427	0.98043430
5.00	0.91251762	1.00226761	0.99224644	0.99218691	0.99218691
5.50	0.94921662	1.00410505	0.99738052	0.99731801	0.99731799
6.00	0.97599696	1.00270216	0.99929900	0.99926011	0.99926011
6.50	0.99285863	1.00090986	0.99738052	0.99986993	0.99986995
7.00	0.99980165	1.00002606	0.99999817	0.99999779	0.99999779
7.10	1.00000000	1.00000000	1.00000000	1.00000000	1.00000000

TABLE II

VELOCITY DISTRIBUTION u/U_1 FOR VARIOUS CURVATURES

η \ A	-0.06	-0.04	-0.02	0.00	+0.02	+0.04	+0.06
0.00	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
0.02	0.0084	0.0078	0.0072	0.0067	0.0062	0.0057	0.0054
0.04	0.0168	0.0156	0.0145	0.0133	0.0123	0.0114	0.0107
0.06	0.0252	0.0235	0.0217	0.0200	0.0184	0.0172	0.0161
0.08	0.0337	0.0313	0.0290	0.0267	0.0246	0.0229	0.0214
0.10	0.0421	0.0392	0.0362	0.0334	0.0308	0.0280	0.0268
0.20	0.0845	0.0785	0.0725	0.0667	0.0615	0.0570	0.0534
0.30	0.1271	0.1179	0.1088	0.1000	0.0921	0.0853	0.0798
0.40	0.1699	0.1575	0.1452	0.1334	0.1226	0.1134	0.1059
0.50	0.2130	0.1972	0.1816	0.1666	0.1530	0.1413	0.1319
0.60	0.2562	0.2369	0.2180	0.1998	0.1832	0.1691	0.1576
0.70	0.2996	0.2767	0.2543	0.2329	0.2134	0.1966	0.1830
0.80	0.3430	0.3165	0.2906	0.2658	0.2433	0.2240	0.2082
0.90	0.3864	0.3562	0.3267	0.2986	0.2730	0.2511	0.2330
1.00	0.4297	0.3957	0.3626	0.3311	0.3025	0.2779	0.2576
1.50	0.6421	0.5886	0.5371	0.4886	0.4445	0.4065	0.3748
2.00	0.8376	0.7651	0.6963	0.6319	0.5734	0.5226	0.4800
2.50	1.0036	0.9145	0.8310	0.7534	0.6830	0.6215	0.5693
3.00	1.1334	1.0301	0.9353	0.8480	0.7690	0.6995	0.6401
3.50	1.2299	1.1131	1.0088	0.9147	0.8301	0.7656	0.6914
4.00	1.3035	1.1706	1.0567	0.9568	0.8684	0.7910	0.7241
4.50	1.3664	1.2129	1.0869	0.9804	0.8887	0.8093	0.7409
5.00	1.4278	1.2482	1.1070	0.9922	0.8962	0.8148	0.7453
5.50	1.4923	1.2816	1.1224	0.9973	0.8958	0.8118	0.7411
6.00	1.5624	1.3157	1.1361	0.9993	0.8913	0.8037	0.7314
6.50	1.6393	1.3513	1.1494	0.9999	0.8846	0.7931	0.7185
7.10	1.7422	1.3966	1.1655	1.0000	0.8757	0.7788	0.7013

TABLE II A

VELOCITY DISTRIBUTION AT SEPARATION

FOR VARIOUS CURVATURES

η \ A	-0.06	0.00	+0.06
0.00	0.00000	0.00000	0.00000
0.05	0.00014	0.00009	0.00006
0.10	0.00058	0.00037	0.00026
0.20	0.00237	0.00149	0.00104
0.30	0.00535	0.00334	0.00233
0.40	0.00955	0.00594	0.00414
0.50	0.01499	0.00928	0.00644
1.00	0.06118	0.03713	0.02530
1.50	0.14027	0.08339	0.05582
2.00	0.25306	0.14759	0.09720
2.50	0.39787	0.22846	0.14840
3.00	0.56890	0.32353	0.20800
3.50	0.75499	0.42870	0.27410
4.00	0.94092	0.53834	0.34427
4.50	1.11207	0.64573	0.41565
5.00	1.26044	0.74439	0.48515
5.50	1.38743	0.82939	0.54980
6.00	1.50084	0.89850	0.60717
6.50	1.64941	0.95224	0.65565
7.00	1.71956	0.99309	0.69460
7.10	1.74216	1.00000	0.70126

TABLE II A

VELOCITY DISTRIBUTION AT SEPARATION

FOR VARIOUS CURVATURES

η \ A	-0.06	0.00	+0.06
0.00	0.00000	0.00000	0.00000
0.05	0.00014	0.00009	0.00006
0.10	0.00058	0.00037	0.00026
0.20	0.00237	0.00149	0.00104
0.30	0.00535	0.00334	0.00233
0.40	0.00955	0.00594	0.00414
0.50	0.01499	0.00928	0.00644
1.00	0.06118	0.03713	0.02530
1.50	0.14027	0.08339	0.05582
2.00	0.25306	0.14759	0.09720
2.50	0.39787	0.22846	0.14840
3.00	0.56890	0.32353	0.20800
3.50	0.75499	0.42870	0.27410
4.00	0.94092	0.53834	0.34427
4.50	1.11207	0.64573	0.41565
5.00	1.26044	0.74439	0.48515
5.50	1.38743	0.82939	0.54980
6.00	1.50084	0.89850	0.60717
6.50	1.64941	0.95224	0.65565
7.00	1.71956	0.99309	0.69460
7.10	1.74216	1.00000	0.70126

TABLE III

TEMPERATURE DISTRIBUTION FOR VARIOUS CURVATURES AT PRANDTL NO=0.1

η \ A	-0.06	0.00	+0.06
0.00	0.0000	0.0000	0.0000
0.02	0.0038	0.0034	0.0032
0.04	0.0077	0.0068	0.0064
0.06	0.0115	0.0102	0.0095
0.08	0.0153	0.0136	0.0127
0.10	0.0192	0.0170	0.0159
0.30	0.0575	0.0511	0.0477
0.50	0.0958	0.0851	0.0795
0.70	0.1341	0.1192	0.1113
0.90	0.1723	0.1532	0.1431
1.50	0.2863	0.2548	0.2382
2.10	0.3984	0.3551	0.3324
2.70	0.5066	0.4532	0.4252
3.30	0.6088	0.5479	0.5159
3.90	0.7025	0.6378	0.6036
4.50	0.7857	0.7218	0.6879
5.10	0.8566	0.7989	0.7679
5.70	0.9144	0.8685	0.8434
6.30	0.9591	0.9302	0.9139
6.90	0.9916	0.9839	0.9793

TABLE III A

TEMPERATURE DISTRIBUTION FOR VARIOUS CURVATURES AT PRANDTL NO=1.0

η \ A	-0.06	0.00	+0.06
0.00	0.0000	0.0000	0.0000
0.02	0.0078	0.0068	0.0058
0.06	0.0235	0.0203	0.0174
0.10	0.0391	0.0338	0.0290
0.50	0.1953	0.1689	0.1450
0.90	0.3493	0.3024	0.2601
1.50	0.5666	0.4941	0.4170
2.10	0.7493	0.6639	0.5802
2.70	0.8798	0.7995	0.7116
3.30	0.9545	0.8945	0.8155
3.90	0.9871	0.9518	0.8907
4.50	0.9974	0.9810	0.9404
5.10	0.9997	0.9937	0.9704
5.70	1.0000	0.9982	0.9869
6.30	1.0000	0.9996	0.9953
6.90	1.0000	1.0000	0.9993
7.10	1.0000	1.0000	1.0000

TABLE III A

TEMPERATURE DISTRIBUTION FOR VARIOUS CURVATURES AT PRANDTL NO=1.0

η \ A	-0.06	0.00	+0.06
0.00	0.0000	0.0000	0.0000
0.02	0.0078	0.0068	0.0058
0.06	0.0235	0.0203	0.0174
0.10	0.0391	0.0338	0.0290
0.50	0.1953	0.1689	0.1450
0.90	0.3493	0.3024	0.2601
1.50	0.5666	0.4941	0.4170
2.10	0.7493	0.6639	0.5802
2.70	0.8798	0.7995	0.7116
3.30	0.9545	0.8945	0.8155
3.90	0.9871	0.9518	0.8907
4.50	0.9974	0.9810	0.9404
5.10	0.9997	0.9937	0.9704
5.70	1.0000	0.9982	0.9869
6.30	1.0000	0.9996	0.9953
6.90	1.0000	1.0000	0.9993
7.10	1.0000	1.0000	1.0000

TABLE IV

TEMPERATURE DISTRIBUTION FOR VARIOUS CURVATURES AT SEPARATION
AT PRANDTL NO=1.0

η \ A	-0.06	0.00	+0.06
0.00	0.0000	0.0000	0.0000
0.04	0.0107	0.0087	0.0075
0.08	0.0213	0.0218	0.0188
0.10	0.0266	0.0218	0.0188
0.50	0.1331	0.1092	0.0938
0.90	0.2395	0.1965	0.1687
1.10	0.2926	0.2401	0.2062
1.50	0.3981	0.3271	0.2809
1.90	0.5018	0.4131	0.3552
2.10	0.5524	0.4556	0.3921
2.70	0.6957	0.5794	0.5008
3.30	0.8179	0.6939	0.6048
3.90	0.9086	0.7937	0.7014
4.50	0.9638	0.8738	0.7874
5.10	0.9894	0.9316	0.8603
5.70	0.9979	0.9685	0.9185
6.30	0.9997	0.9888	0.9621
6.90	1.0000	0.9983	0.9924
7.10	1.0000	1.0000	1.0000

TABLE IV A

TEMPERATURE DISTRIBUTION FOR VARIOUS CURVATURES AT SEPARATION AT PRANDTL NO=5.0

η \ A	-0.06	0.00	+0.06
0.00	0.0000	0.0000	0.0000
0.04	0.0157	0.0133	0.0114
0.08	0.0315	0.0265	0.0224
0.10	0.0394	0.0332	0.0285
0.50	0.1968	0.1657	0.1424
0.90	0.3535	0.2980	0.2561
1.10	0.4310	0.3636	0.3128
1.50	0.5812	0.4927	0.4247
1.90	0.7190	0.6158	0.5335
2.10	0.7801	0.6737	0.5858
2.70	0.9186	0.8235	0.7299
3.30	0.9827	0.9257	0.8458
3.90	0.9984	0.9778	0.9259
4.50	1.0000	0.9958	0.9713
5.10	1.0000	0.9995	0.9914
5.70	1.0000	1.0000	0.9981
6.30	1.0000	1.0000	0.9991
6.90	1.0000	1.0000	1.0000
7.10	1.0000	1.0000	1.0000

TABLE V

TEMPERATURE GRADIENTS AT SURFACE AT VARIOUS PRANDTL NUMBERS
FOR VARIOUS CURVATURES

A	-0.06	-0.04	-0.02	0.00	+0.02	+0.04	+0.06
η \ $f''(0)$	0.4197	0.3907	0.3617	0.3336	0.3080	0.2863	0.2690
0.1	0.1916	0.1830	0.1760	0.1703	0.1657	0.1620	0.1591
0.2	0.2320	0.2190	0.2075	0.1976	0.1893	0.1824	0.1769
0.3	0.2639	0.2489	0.2350	0.2223	0.2112	0.2018	0.1940
0.4	0.2901	0.2741	0.2588	0.2443	0.2312	0.2198	0.2102
0.5	0.3123	0.2958	0.2796	0.2640	0.2495	0.2366	0.2256
0.6	0.3304	0.3139	0.2976	0.2816	0.2665	0.2528	0.2407
0.7	0.3477	0.3309	0.3141	0.2976	0.2818	0.2672	0.2543
0.8	0.3634	0.3463	0.3292	0.3122	0.2958	0.2807	0.2671
0.9	0.3779	0.3605	0.3430	0.3256	0.3088	0.2932	0.2791
1.0	0.3913	0.3736	0.3558	0.3381	0.3210	0.3049	0.2903
2.0	0.4925	0.4725	0.4523	0.4320	0.4123	0.3938	0.3770
3.0	0.5637	0.5420	0.5200	0.4978	0.4763	0.4562	0.4379
4.0	0.6205	0.5974	0.5739	0.5503	0.5273	0.5059	0.4865
5.0	0.6686	0.6444	0.6196	0.5947	0.5705	0.5479	0.5276
6.0	0.7108	0.6856	0.6597	0.6336	0.6083	0.5847	0.5636
7.0	0.7487	0.7225	0.6956	0.6685	0.6422	0.6177	0.5958
8.0	0.7881	0.7561	0.7283	0.7003	0.6731	0.6478	0.6252

TABLE V A

TEMPERATURE GRADIENTS AT SURFACE AT VARIOUS PRANDTL NUMBERS
FOR VARIOUS CURVATURES
AT SEPERATION

η \ A	-0.06	-0.04	-0.02	0.00	+0.02	+0.04	+0.06
0.1	0.1640	0.1582	0.1540	0.1510	0.1488	0.1472	0.1461
0.2	0.1835	0.1736	0.1661	0.1605	0.1564	0.1534	0.1512
0.3	0.1997	0.1873	0.1773	0.1696	0.1638	0.1594	0.1562
0.4	0.2134	0.1993	0.1874	0.1780	0.1707	0.1652	0.1611
0.5	0.2250	0.2099	0.1967	0.1859	0.1774	0.1708	0.1658
0.6	0.2352	0.2193	0.2052	0.1933	0.1837	0.1762	0.1704
0.7	0.2441	0.2278	0.2130	0.2002	0.1897	0.1814	0.1749
0.8	0.2522	0.2355	0.2202	0.2067	0.1954	0.1864	0.1792
0.9	0.2595	0.2425	0.2268	0.2127	0.2009	0.1912	0.1834
1.0	0.2663	0.2490	0.2329	0.2184	0.2060	0.1958	0.1875
2.0	0.3151	0.2960	0.2780	0.2614	0.2465	0.2333	0.2219
3.0	0.3477	0.3274	0.3082	0.2905	0.2745	0.2602	0.2476
4.0	0.3729	0.3516	0.3315	0.3130	0.2962	0.2812	0.2680
5.0	0.3937	0.3716	0.3508	0.3315	0.3141	0.2986	0.2849
6.0	0.4116	0.3883	0.3673	0.3475	0.3295	0.3135	0.2994
7.0	0.4274	0.4040	0.3819	0.3615	0.3431	0.3267	0.3122
8.0	0.4416	0.4177	0.3950	0.3741	0.3553	0.3385	0.3237
9.0	0.4546	0.4301	0.4070	0.3856	0.3664	0.3493	0.3342
10.0	0.4665	0.4415	0.4180	0.3962	0.3766	0.3592	0.3438

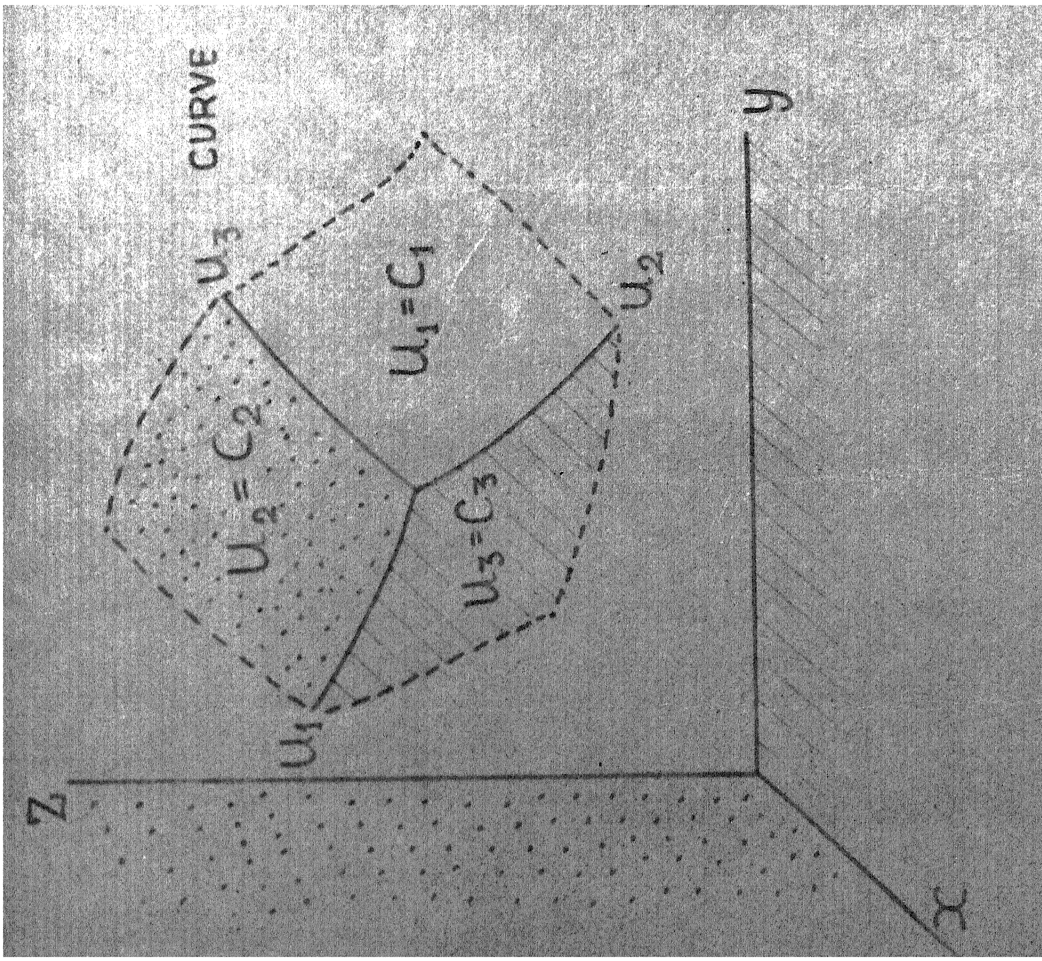

FIGURE -3

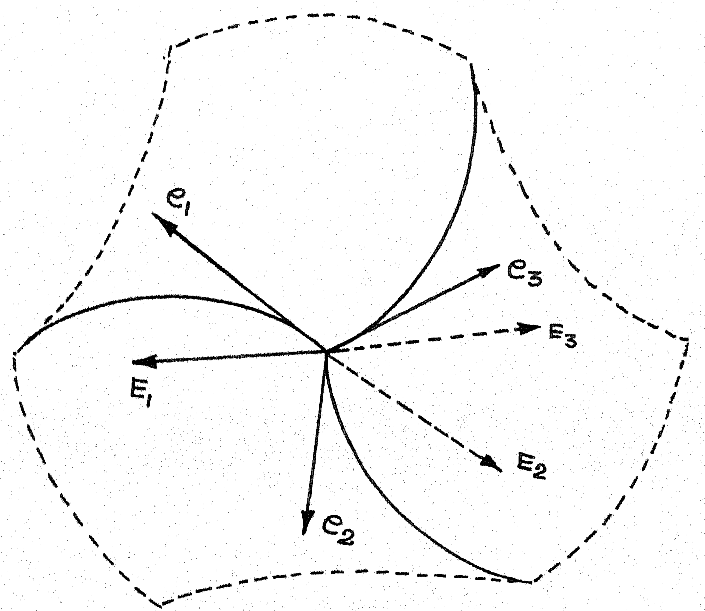

FIGURE -4

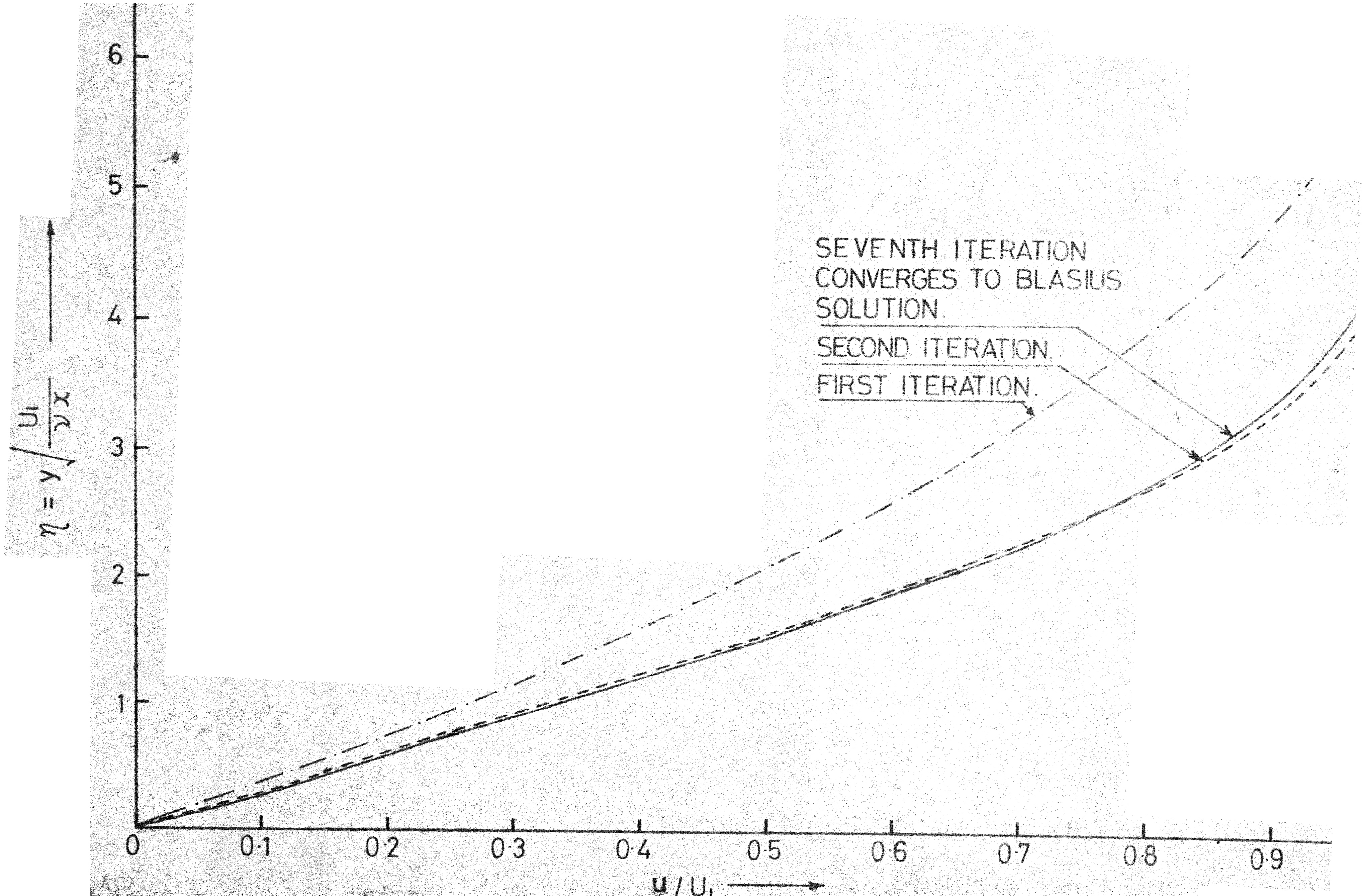

FIG.5 VELOCITY PROFILES AT VARIOUS ITERATIONS FOR A FLAT PLATE (A=0)

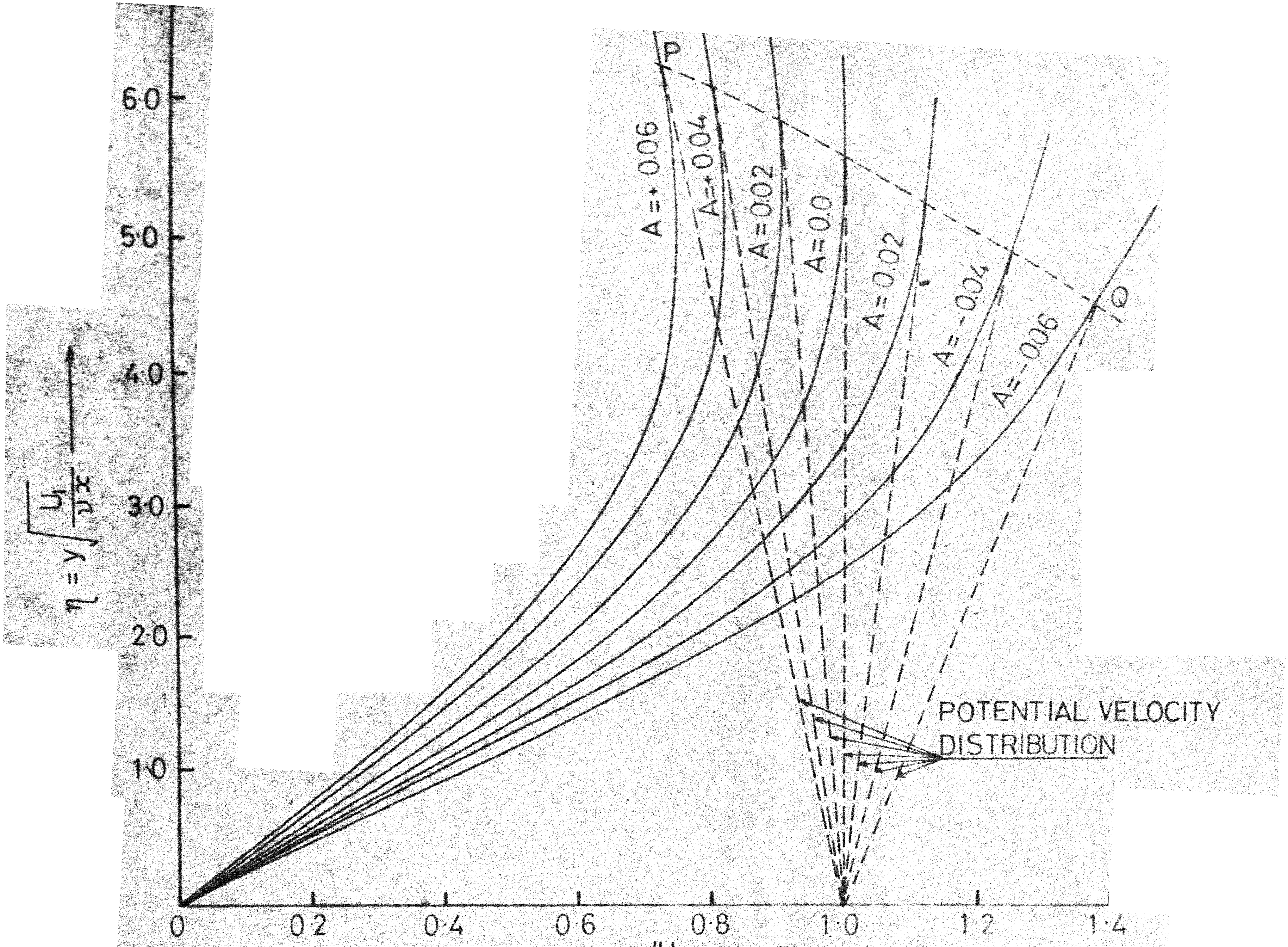

FIG.6. VELOCITY DISTRIBUTION FOR CURVED SURFACES IN THE BOUNDARY LAYER REGION.

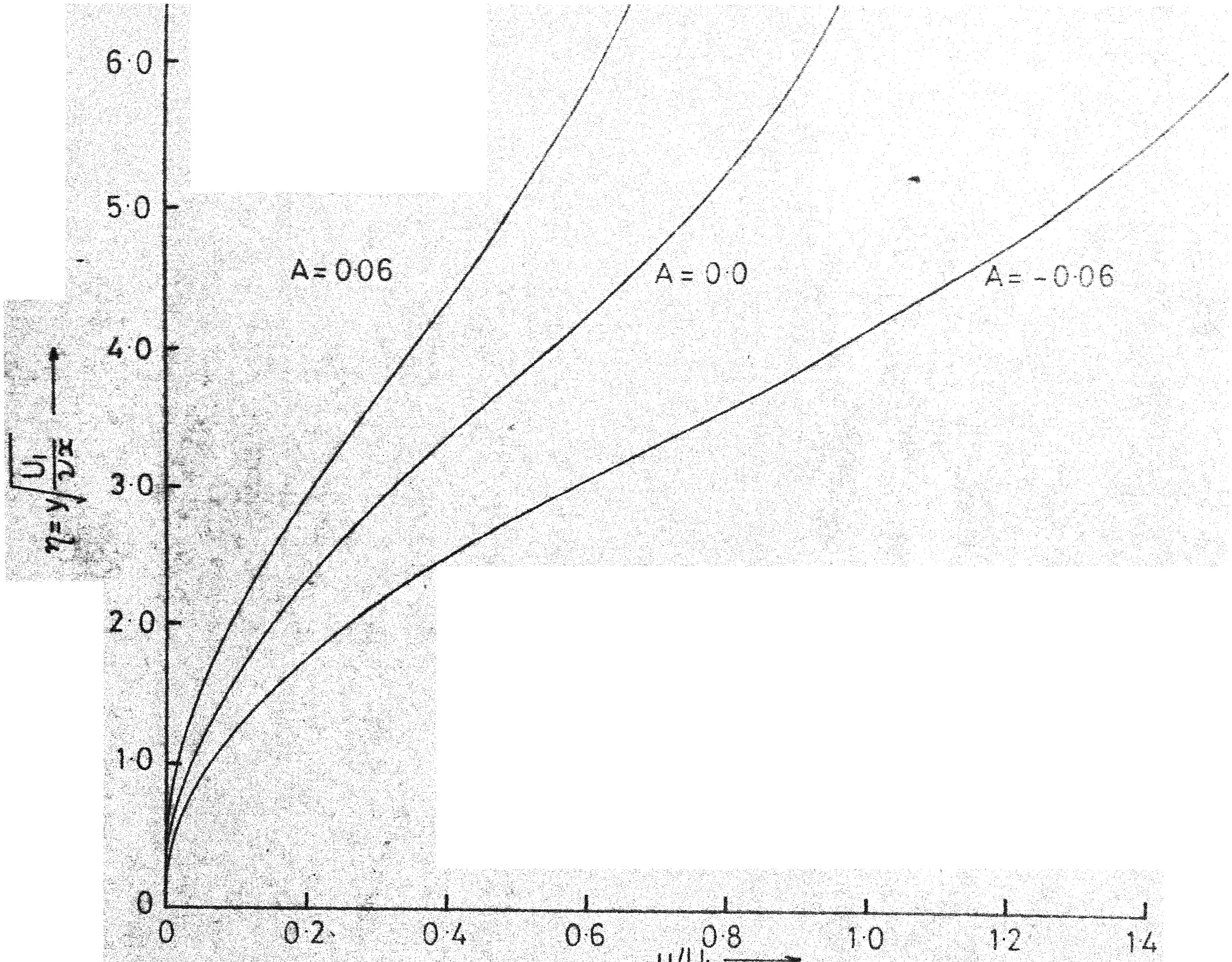

FIG. 7. Velocity distribution for curve surfaces at separation.

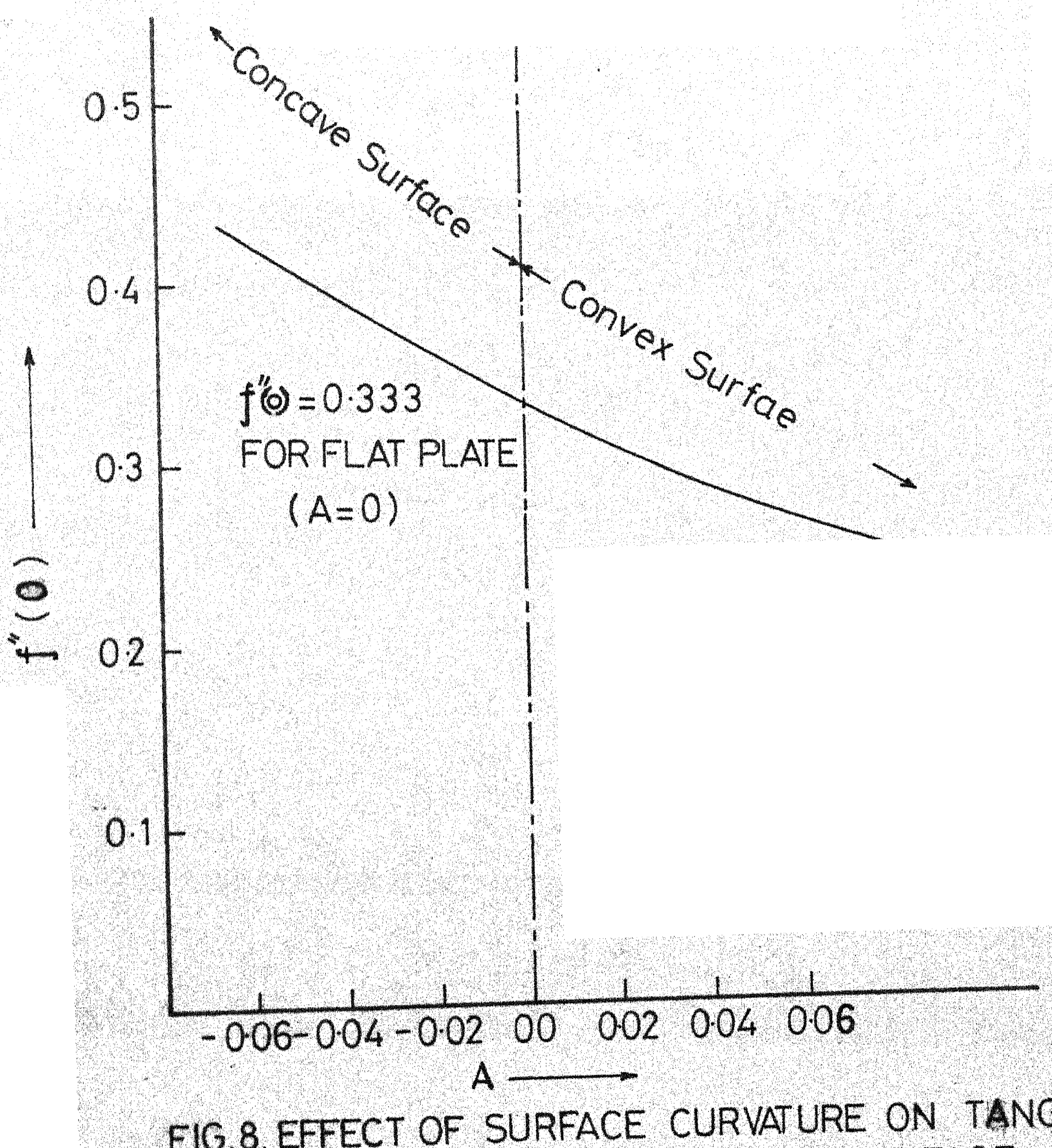

FIG.8. EFFECT OF SURFACE CURVATURE ON TANGENTIAL VELOCITY GRADIENT AT THE SURFACE.

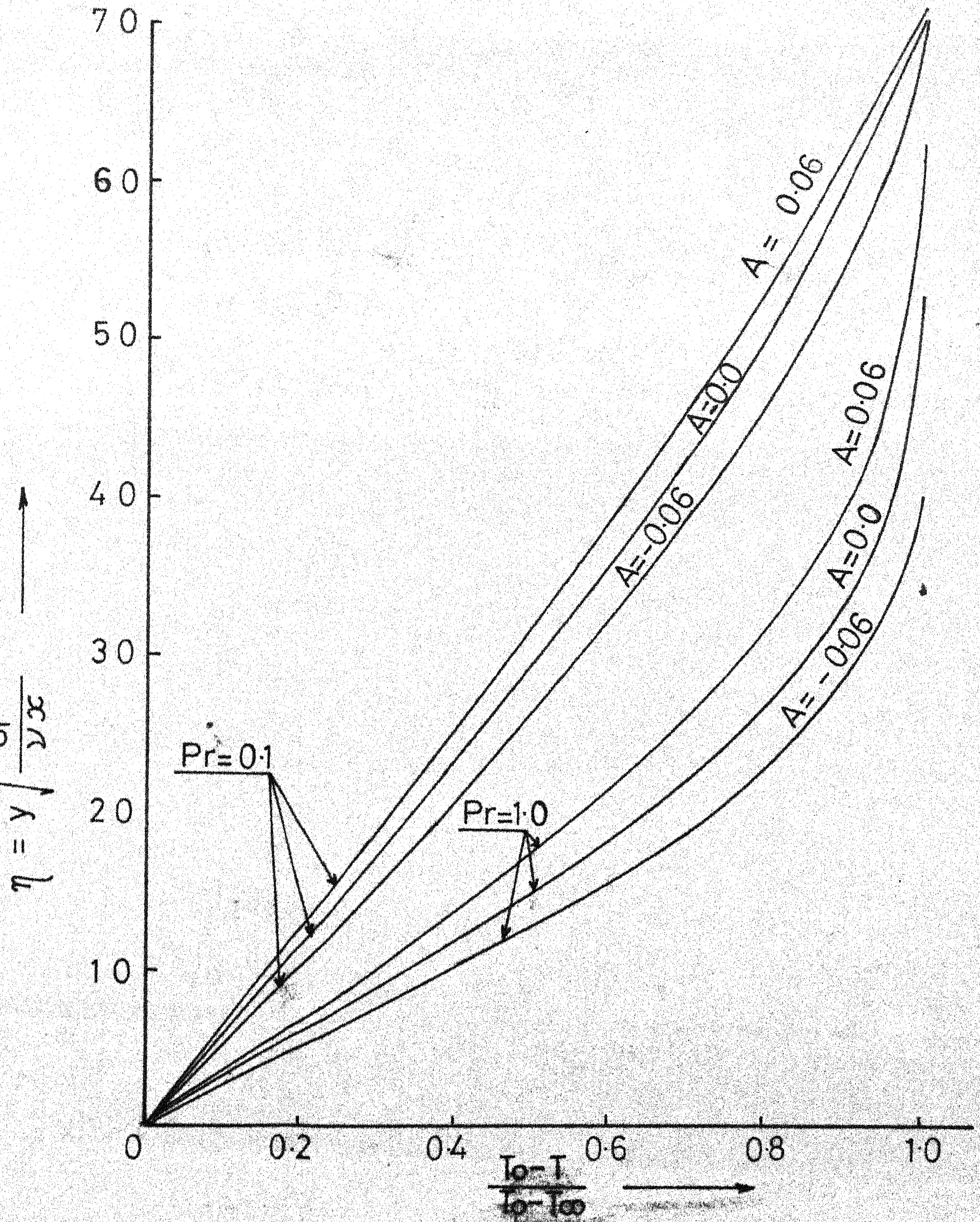

FIG.9. TEMPERATURE DISTRIBUTION FOR CURVED SURFACES IN THE BOUNDARY LAYER REGION.

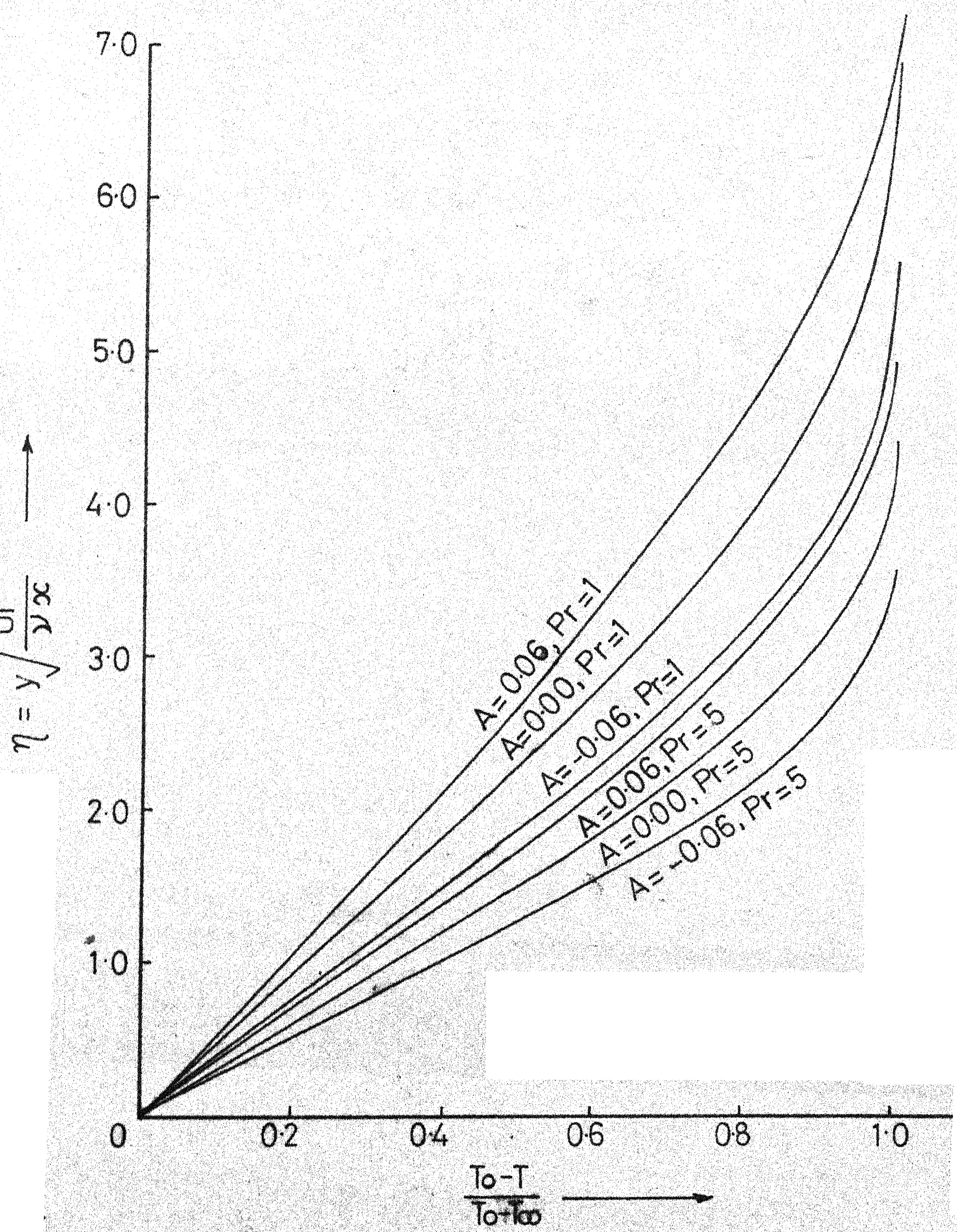

FIG.10. TEMPERATURE DISTRIBUTION FOR CURVED SURFACES AT SEPARATION.

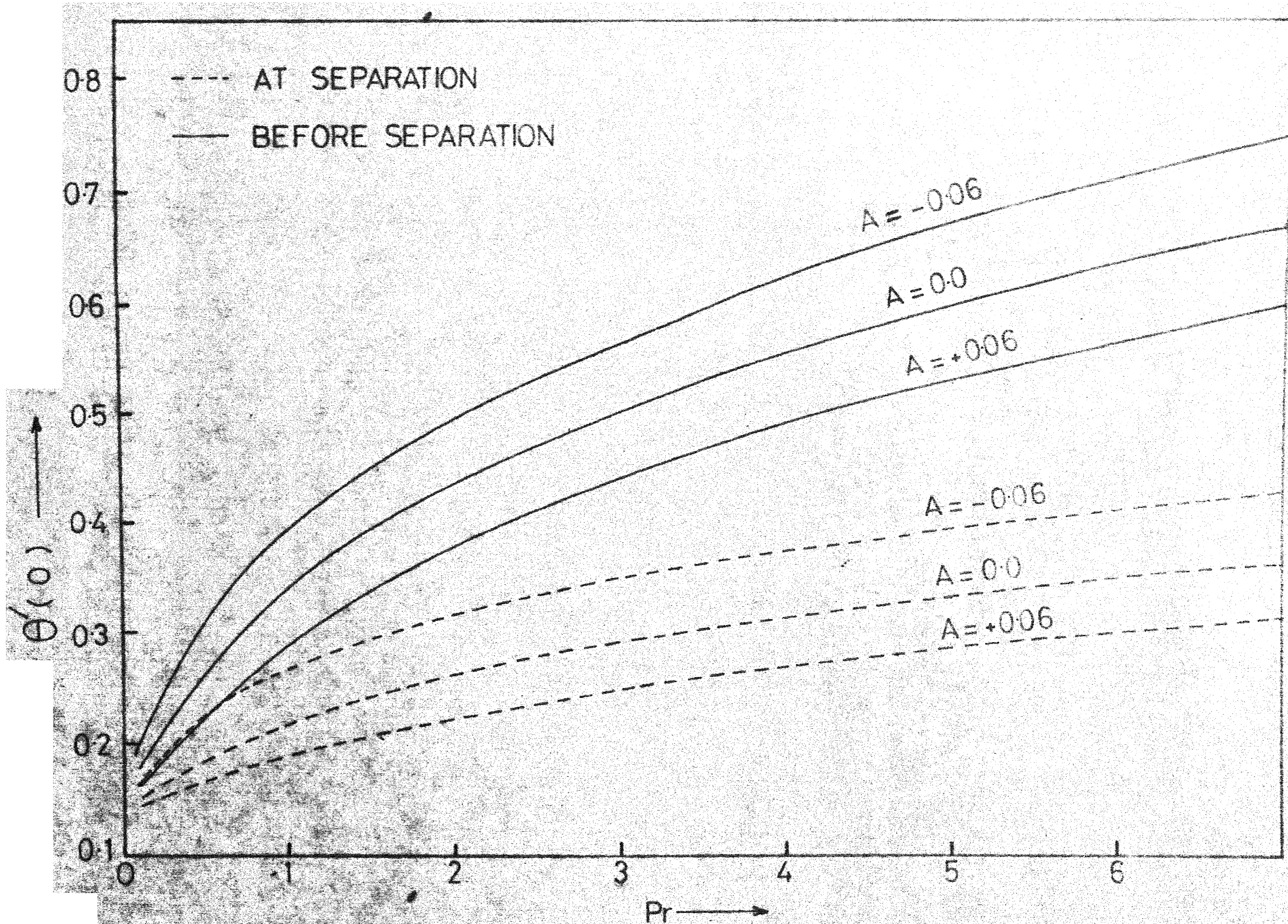

FIG.11. Effect of Prandtl Number on temperature gradient at the surface

APPENDIX A

CURVILINEAR COORDINATES

1. Transformation of Coordinates

Let the rectangular coordinates (x,y,z) of any point be expressed as function of (u_1,u_2,u_3) so that

$$
\begin{aligned}
x &= x\,(u_1,u_2,u_3)\\
y &= y\,(u_1,u_2,u_3)\\
z &= z\,(u_1,u_2,u_3)
\end{aligned}
\tag{1.1}
$$

We can also write u_1,u_2,u_3 as function of (x,y,z)

$$
\begin{aligned}
u_1 &= u_1(x,y,z)\\
u_2 &= u_2(x,y,z)\\
u_3 &= u_3(x,y,z)
\end{aligned}
\tag{1.2}
$$

In order that correspondence between (u_1,u_2,u_3) and (x,y,z) is unique, the functions in equation (1.1), (1.2) have, to be single valued and to have continuous derivatives. In practice, the assumptions may not apply at certain points and special consideration is required.

Thus given a point P with rectangular co-ordinate (x,y,z), we can associate a unique set of coordinate (u_1,u_2,u_3)

called the curvilinear coordinates of P. The set of equations (1.1) and (1.2) define a transformation of coordinates.

2. Orthogonal Curvilinear Coordinates:

The surfaces $u_1 = c_1$, $u_2 = c_2$, $u_3 = c_3$ where c_1, c_2, c_3 are constants, are called coordinate surfaces and each pair of these surfaces intersect in curves, called coordinate curves or line (see Figure-3). If the coordinate surfaces intersect at right angles, the curvilinear coordinate system is called orthogonal. The u_1, u_2, u_3 coordinate curves of curvilinear system are analogous to x, y, z coordinate axes of a rectangular system.

3. Unit Vectors in Curvilinear Systems:

Let $r = xi + yj + zk$ be the position vector of the point P. Then equation (1.1) can be written as :

$$r = r(u_1, u_2, u_3)$$

A tangent vector to curve u_1 at P (for which u_2 and u_3 are constant) is $\frac{\partial r}{\partial u_1}$. Then a unit tangent vector in this direction is :

$$e_1 = \frac{\partial r}{\partial u_1} \Big/ \left|\frac{\partial r}{\partial u_1}\right|$$

so that:

$$\frac{\partial r}{\partial u_1} = h_1 e_1$$

where:

$$h_1 = \left|\frac{\partial r}{\partial u_1}\right|$$

Similarly, if e_2 and e_3 are unit tangent vectors to u_2 and u_3 curves respectively at P, then:

$$\frac{\partial r}{\partial u_2} = h_2 e_2$$

and $$\frac{\partial r}{\partial u_3} = h_3 e_3$$

where $h_2 = \left|\frac{\partial r}{\partial u_2}\right|$ and $h_3 = \left|\frac{\partial r}{\partial u_3}\right|$

The quantities h_1, h_2, h_3 are called scale factors. The unit vectors e_1, e_2, e_3 are in the direction of increasing u_1, u_2, u_3 respectively.

Since u_1 is a vector at P normal to the surface $u_1 = c_1$, a unit vector in this direction is given by :

$$E_1 = \nabla u_1 \Big/ \left|\nabla u_1\right|$$

Similarly unit vectors

$$E_2 = \nabla u_2 \Big/ \left|\nabla u_2\right|$$

and

$$E_3 = \nabla u_3 \Big/ \left|\nabla u_3\right|$$

at P are normal to the surfaces $u_2 = c_2$ and $u_3 = c_3$ respectively.

Thus at each point P of a curvilinear system, there exist, in general, two sets of unit vectors e_1, e_2, e_3 tangent to the coordinate curves (see Figure-4) and E_1, E_2, E_3 normal to the coordinate surfaces. The sets become identical if and only if the curvilinear coordinate system is orthogonal. Both sets are analogous to the i, j, k unit vectors in rectangular

co-ordinates but are unlike them in that they may change direction from point to point.

A vector $\overline{A}$ can be represented in terms of the unit base vectors e_1, e_2, e_3 or E_1, E_2, E_3 in the form :

$$\overline{A} = A_1 e_1 + A_2 e_2 + A_3 e_3$$

or

$$\overline{A} = a_1 E_1 + a_2 E_2 + a_3 E_3$$

where A_1, A_2, A_3 and a_1, a_2, a_3 are the respective components of $\overline{A}$ in each system.

4. <u>Arc Length and Volume Elements</u>:

From $r = r(u_1, u_2, u_3)$ we have

$$dr = \frac{\partial r}{\partial u_1} du_1 + \frac{\partial r}{\partial u_2} du_2 + \frac{\partial r}{\partial u_3} du_3$$

$$= h_1 du_1 e_1 + h_2 du_2 e_2 + h_3 du_3 e_3$$

Differential arc length ds is determined from $ds^2 = dr.dr$.

For orthogonal systems :

$$e_1 . e_2 = e_2 . e_3 = e_3 . e_1 = 0$$

hence :

$$ds^2 = h_1^2 du_1^2 + h_2^2 du_2^2 + h_3^2 du_3^2$$

Along u_1 curve, u_2 and u_3 are constant

hence :

$$dr = h_1 du_1 e_1$$

Then the differential of arc length ds_1 along u_1 at P is $h_1 du_1$.

Similarly, the differential arc length along u_2 and u_3 at P are :

$$ds_2 = h_2 du_2$$

$$ds_3 = h_3 du_3$$

The volume element for an orthogonal curvilinear coordinate system is given by:

$$dV = h_1 h_2 h_3 du_1 du_2 du_3, \quad |e_1 . e_2 \times e_3| = 1$$

5. $\nabla\Phi$ and $\nabla^2\Phi$ in Orthogonal Curvilinear Co-ordinates:

$$dr = h_1 du_1 e_1 + h_2 du_2 e_2 + h_3 du_3 e_3 \tag{5.1}$$

and let

$$\nabla\Phi = f_1 e_1 + f_2 e_2 + f_3 e_3 \tag{5.2}$$

where f_1, f_2, f_3 are to be determined.

$$d\Phi = \nabla\Phi . dr$$

$$= h_1 f_1 du_1 + h_2 f_2 du_2 + h_3 f_3 du_3 \tag{5.3}$$

Using equations (5.1) and (5.2), equation (5.3) has been obtained.

Also $$d\Phi = \frac{\partial\Phi}{\partial u_1} du_1 + \frac{\partial\Phi}{\partial u_2} du_2 + \frac{\partial\Phi}{\partial u_3} du_3 \tag{5.4}$$

Equations (5.3) and (5.4) give :

$$f_1 = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} \; ; \; f_2 = \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2} \; ; \; f_3 = \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3}$$

Substituting these values in equation (5.2), we get:

$$\nabla\Phi = \frac{1}{h_1}\frac{\partial\Phi}{\partial u_1} e_1 + \frac{1}{h_2}\frac{\partial\Phi}{\partial u_2} e_2 + \frac{1}{h_3}\frac{\partial\Phi}{\partial u_3} e_3 \tag{5.5}$$

This indicates the operator equivalence :

$$\nabla = \frac{e_1}{h_1}\frac{\partial}{\partial u_1} + \frac{e_2}{h_2}\frac{\partial}{\partial u_2} + \frac{e_3}{h_3}\frac{\partial}{\partial u_3}$$

Similarly[7], Div $\bar{A}$, Curl $\bar{A}$ and Laplacian of Φ can also be derived in terms of curvilinear co-ordinate system.

Expressions for these are given below :

$$\text{Div } \bar{A} = \nabla \cdot \bar{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(A_2h_1h_3) + \frac{\partial}{\partial u_3}(A_3h_1h_2) \right] \tag{5.6}$$

$$\text{Curl } \bar{A} = \nabla \times \bar{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} h_1e_1 & h_2e_2 & h_3e_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ A_1h_1 & A_2h_2 & A_3h_3 \end{vmatrix} \tag{5.7}$$

$$\text{Laplacian of } \Phi = \nabla^2\Phi$$

$$= \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}\frac{h_2h_3}{h_1}\frac{\partial\Phi}{\partial u_1} + \frac{\partial}{\partial u_2}\frac{h_1h_3}{h_2}\frac{\partial\Phi}{\partial u_2} + \frac{\partial}{\partial u_3}\frac{h_1h_2}{h_3}\frac{\partial\Phi}{\partial u_3} \right] \tag{5.8}$$

APPENDIX B

VORTICITY

Average angular velocities along x,y and z axes are :

$$\omega_x = \frac{1}{2} \quad \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$

$$\omega_y = \frac{1}{2} \quad \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$$

$$\omega_z = \frac{1}{2} \quad \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Therefore, angular velocity ω of a fluid element in terms of the velocity field is

$$\omega = \omega_x\, i + \omega_y\, j + \omega_z\, k$$

$$= \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) i + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) j + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) k\right]$$

ω is written as :

$$\omega = \frac{1}{2}\,\Omega$$

where Ω is called vorticity vector. If a line is drawn in the fluid so that the tangent to it at each point is in the direction of the vorticity vector Ω at that point, the line is called vortex line. The angular motion of fluid elements

is a physical action and does not depend on co-ordinate system. It can also be written as:

$$\omega = \frac{1}{2}\ \mathrm{Curl}\ \bar{V}$$

For irrotational flow: $\mathrm{Curl}\ \bar{V} = 0$

<u>Expressions for Vorticity in Orthogonal Curvilinear Co-ordinate System</u> :

$$\Omega_1 = \frac{1}{h_2h_3}\left[\frac{\partial}{\partial u_2}(h_3q_3) - \frac{\partial}{\partial u_3}(h_2q_2)\right]$$

$$\Omega_2 = \frac{1}{h_1h_3}\left[\frac{\partial}{\partial u_3}(h_1q_1) - \frac{\partial}{\partial u_1}(h_3h_3)\right]$$

$$\Omega_3 = \frac{1}{h_1h_2}\left[\frac{\partial}{\partial u_1}(h_2q_2) - \frac{\partial}{\partial u_2}(h_1q_1)\right]$$

where q_1, q_2, q_3 are components of velocity in directions u_1, u_2 and u_3 respectively ; h_1, h_2, h_3 have the meanings given in Appendix-A. In two dimensional flow vorticity is Ω_3.

APPENDIX C

NUMERICAL SOLUTION OF DIFFERENTIAL EQUATIONS

Whenever an explicit solution of a differential equation is possible, it is usually best to use this explicit solution rather than to resort to numerical methods. Unfortunately, in many cases where the solution of a differential equation is needed it proves to be impossible to obtain a solution in explicit form.

There are several methods of numerical solution of differential equations of first order. These methods can be extended to solve equations of higher order also. There are some special methods to solve higher order equations. In all above methods the necessary initial conditions (conditions at the start of numerical integration) are equal to the order of differential equations. When few conditions are given at the start and the remaining ones at some other point, the problem is called a Two - Point Boundary Value Problem. The equation may be linear or non-linear.

Solution of Linear Equations:

The procedure of linear equation is considerably simpler than that of non-linear.

The general solution of an equation of the type:

$$y'' + Py' + Qy = R$$

with P, Q, and R as functions of x has the form:

$$y = Au + Bv + w$$

where A and B are constants, u and v are independent solutions of the equation obtained by setting $R = 0$, and w is a particular solution of the original equation. If one condition is available at the starting point, the number of arbitrary constants reduces by one. Effectively, the solution now has the form:

$$y = Au + w$$

where u and w are chosen so that y satisfies the initial condition regardless of A. The solutions u and w can be determined numerically by any suitable step by step process, and A can be finally determined by satisfying the terminal condition.

Solution of Non-Linear Equations:

For non-linear equations, the above method fails as P, Q and R are functions not of independent variable only. The simplest way to tackle the two - end - point problem for non-linear equations is to start with assumed conditions at the initial point and calculate the trial solution as for as the end point. By repetition of this scheme of trial and correction when second boundary condition is satisfied, the final solution is obtained. This method is

suitable for well behaved functions. For a system of n equations in n unknowns, the difficulties increase with n. This method is also known as " Garden Hose " method.

Another method is Linearization and iteration where the non-linear terms are all grouped together. Each time, a linear non - homogeneous equation is solved. It is considered better than the first method.

<u>Quasilinearisation and Iteration</u> :

Quasilinearization may be viewed as an extension of Newton's method for the solution of algebraic equations to solution of differential equations.

Consider a non-linear equation

$$g(x,y,y',y'') = y'' + F(x,y,y') = 0 \qquad \text{(C-1)}$$

with boundary conditions

$$y(a) = a_1$$
$$y(b) = a_2$$

Suppose (y_o, y_o', y_o'') is an approximate solution of the above equation. Making approximation in functional space we can write :

$$g(x,y,y',y'') = g(x,y_o,y_o',y_o'') + (y - y_o)\frac{\partial g}{\partial y} + (y' - y_o')\frac{\partial g}{\partial y'}$$

$$+ (y'' - y_o'')\frac{\partial g}{\partial y''} + \text{higher order terms.}$$

If y_0, y'_0, y''_0 are very close to true solution, higher order terms may be neglected.

Hence:

$$g(x,y,y',y'') = g(x,y_0,y'_0,y''_0) + (y - y_0)\left(\frac{\partial g}{\partial y}\right)_0 + (y' - y'_0)\left(\frac{\partial g}{\partial y'}\right)_0 + (y'' - y''_0)\left(\frac{\partial g}{\partial y''}\right)_0 \qquad \text{(C-2)}$$

From equation (C-1)

$$\frac{\partial g}{\partial y} = \frac{\partial F}{\partial y}$$

$$\frac{\partial g}{\partial y'} = \frac{\partial F}{\partial y'} \qquad \text{and}$$

$$\frac{\partial g}{\partial y''} = 1.0$$

so equation (C-2) can be written as:

$$y'' + F(x,y,y') = y''_0 + F(x,y_0,y'_0) + (y - y_0)\left(\frac{\partial F}{\partial y}\right)_0 + (y' - y'_0)\left(\frac{\partial F}{\partial y'}\right)_0 + (y'' - y''_0)\left(\frac{\partial F}{\partial y''}\right)_0$$

$$= 0.0 \qquad \text{(C-3)}$$

Therefore :

$$y'' + F(x,y,y') = y''_0 + F(x,y_0,y'_0) + (y - y_0)\left(\frac{\partial F}{\partial y}\right)_0 + (y' - y'_0)\left(\frac{\partial F}{\partial y'}\right)_0 + y'' - y''_0 = 0$$

or $\quad y'' + (y - y_0)\left(\frac{\partial F}{\partial y}\right)_0 + (y' - y_0')\left(\frac{\partial F}{\partial y'}\right)_0 + F(x, y_0, y_0') = 0$

(C-4)

This is a linear differential equation with non - homogeneous term as :

$$F(x, y_0, y_0') - y_0\left(\frac{\partial F}{\partial y}\right)_0 - y_0'\left(\frac{\partial F}{\partial y'}\right)_0$$

In general, if y_n is the n^{th} approximation, equation (C-4) can be written as :

$$y''_{n+1} + (y'_{n+1} - y'_n)\left(\frac{\partial F}{\partial y'}\right)_n + (y_{n+1} - y_n)\left(\frac{\partial F}{\partial y}\right)_n + F(x, y_n, y_n') = 0 \qquad \text{(C-5)}$$

or

$$y''_{n+1} + \left(\frac{\partial F}{\partial y'}\right)_n y'_{n+1} + \left(\frac{\partial F}{\partial y}\right)_n y_{n+1} = -F(x, y_n, y_n') + y_n\left(\frac{\partial F}{\partial y}\right)_n + y_n'\left(\frac{\partial F}{\partial y'}\right)_n \qquad \text{(C-6)}$$

Equation (C-6) has to satisfy the boundary conditions of the original equation that is :

$$y_{n+1}(o) = a_1$$

$$y_{n+1}(b) = a_2 \text{ , (outer boundary condition)}$$

The equation (C-6) is linear and usual methods for solving linear ordinary differential equation may be used.

Particular solution (Z2) is obtained by integrating the equation (C-6) retaining non - homogeneous terms and using the following conditions :

$y \quad (o) = a_1$

$y' \quad (o) = 0 \quad$ (assumption)

One complementary solution is obtained for each missing initial condition. In the present example, only one initial condition is missing ; hence only one complementary solution (Z1) is to be obtained. We use the following conditions :

$y \quad (o) = 0$

$y' \quad (o) = 1 \quad$ (assumption)

Complete solution is represented as the linear combination of Z1 and Z2 that is $y = C_1 \times Z1 + Z2$

Constant C_1 is obtained by satisfying outer boundary condition. This gives :

$$C_1 = \frac{a_2 - Z2\ (b)}{Z1\ (b)}$$

Making use of C_1, complete solution is obtained.

The assumed (y_o, y_o', y_o'') solution is now replaced by the present (new) solution, and the process may be repeated till desired accuracy is achieved. Each time a linear non - homogeneous equation is solved. Initial value integration may be performed by any suitable method.

In general, if there are p unknown initial conditions, p complementary solutions and one particular integral are obtained. Linear combination of $(p+1)$ solutions gives the complete solution.

$$y = \sum_{i=1}^{p} C_i Z_i + Z_{PI}$$

Number of constants are equal to the number of complementary solutions and are evaluated by satisfying p boundary conditions at the other end. Convergence of this method is more rapid than " Garden Hose Method ".

APPENDIX D

COMPUTER PROGRAMME

```
$JOB MEG147 , TIME008,PAGES 010 ,NAME K N SRIVASTAVA
$IBJOB          MAP
$IBFTC MAIN     NODECK
      DIMENSION Y(91, 5,4),Z2(91, 5,4),Z1(91, 5,4),X(91),Y1(91),
     1    Y2(91),Y3(91),Y4(91),Y5(91)        ,Z3(91, 5,4)
     2  ,P(91, 5,2),ZZ2(91, 5,2),ZZ1(91, 5,2),P1(91),P2(91),P3(91)
C     N=NUMBER OF STATION POINTS
C     ETA INFINITY TAKEN EQUAL TO  7.1
      EQUIVALENCE (Z1,ZZ1        ),(Z2,ZZ2)
      N=81
      A=-0.06
      DO 36 III=1,7
      A=A+0.02
      PRINT11,A
11    FORMAT(2X,F8.2)
C     ASSUMED  SOLUTION
      DO 600 I=1,N
      Y(I,1,1)=0.0
      Y(I,1,2)=0.0
      Y(I,1,3)=0.0
      Y(I,1,4)=0.0
600   CONTINUE
170   K=0
      K=K+1
C     COMPL  SOLUTIONS  Z1   Z2
      Y1(1)=0.0
      Y2(1)=0.0
      Y3(1)=0.0
      Y4(1)=1.0
      Y5(1)=0.0
      DO 100 I=2,N
      IF(I.GT.11) GO TO 222
      H=0.01
      GO TO 333
222   H=0.1
333   CALL RUNGA(Y1(I-1),Y2(I-1),Y3(I-1),Y4(I-1),Y5(I-1),0.0,Y(I,K
     1  Y(I,K,2),Y(I,K,3),Y(I,K,4),Z1,X,I,A,K+1,H )
      Y1(I)=X(I)
      Y2(I)=Z1(I,K+1,1)
      Y3(I)=Z1(I,K+1,2)
      Y4(I)=Z1(I,K+1,3)
      Y5(I)=Z1(I,K+1,4)
100   CONTINUE
      Y1(1)=0.0
      Y2(1)=0.0
      Y3(1)=0.0
      Y4(1)=0.0
      Y5(1)=1.0
```

```
      DO 200 I=2,N
      IF(I.GT.11) GO TO 444
      H=0.01
      GO TO 555
444   H=0.1
555   CALL RUNGA(Y1(I-1),Y2(I-1),Y3(I-1),Y4(I-1),Y5(I-1),0.0,Y(I,K,1),
     1    Y(I,K,2),Y(I,K,3),Y(I,K,4),Z2,X,I,A,K+1,H)
      Y1(I)=X(I)
      Y2(I)=Z2(I,K+1,1)
      Y3(I)=Z2(I,K+1,2)
      Y4(I)=Z2(I,K+1,3)
      Y5(I)=Z2(I,K+1,4)
200   CONTINUE
C     PARTICULAR  INTEGRAL
      Y5(1)=0.0
      DO 400 I=2,N
      IF(I.GT.11) GO TO 666
      H=0.01
      GO TO 777
666   H=0.1
777   CALL RUNGA(Y1(I-1),Y2(I-1),Y3(I-1),Y4(I-1),Y5(I-1),1.0,Y(I,K,1),
     1    Y(I,K,2),Y(I,K,3),Y(I,K,4),Z3,X,I,A,K+1,H)
      Y1(I)=X(I)
      Y2(I)=Z3(I,K+1,1)
      Y3(I)=Z3(I,K+1,2)
      Y4(I)=Z3(I,K+1,3)
      Y5(I)=Z3(I,K+1,4)
400   CONTINUE
      Y(1,K,1)=0.0
C     USE OF  THIRD AND FOURTH BOUNDARY CONDITIONS
      B=1.0/(1.0+A*7.1)
      BB=-A/((1.0+A*7.1)**2)
      Z1P=Z1(N,K+1,2)
      Z2P=Z2(N,K+1,2)
      Z3P=Z3(N,K+1,2)
      Z1DP=Z1(N,K+1,3)
      Z2DP=Z2(N,K+1,3)
      Z3DP=Z3(N,K+1,3)
C
C     CALCULATION OF CONSTANTS
C
      C1=(B*Z2DP-BB*Z2P-Z3P*Z2DP+Z3DP*Z2P)/(Z1P*Z2DP-Z1DP*Z2P)
      C2=(B-Z3P-C1*Z1P)/Z2P
      DO 300 I=2,N
      DO 300 J=1,4
      Y(I,K+1,J)=C1*Z1(I,K+1,J)+C2*Z2(I,K+1,J)+Z3(I,K+1,J)
300   CONTINUE
C     TESTING THE ACCURACY
      Y(1,K+1,1)=0.0
      Y(1,K+1,2)=0.0
      Y(1,K+1,3)=C1
      Y(1,K+1,4)=C2
```

```
      K1=K+1
      PRINT 211 ,(Y(M,K1,2),M=1,N)
211   FORMAT (12X,10(F12.8))
      DELTA=0.0000001
      I=2
180   IF(ABS(Y(I,K+1,2)-Y(I,K,2)).LT.DELTA) GO TO 99
      DO 77 I=1,N
      Y(I,K,1)=Y(I,K+1,1)
      Y(I,K,2)=Y(I,K+1,2)
      Y(I,K,3)=Y(I,K+1,3)
      Y(I,K,4)=Y(I,K+1,4)
77    CONTINUE
      GO TO 170
99    I=I+1
      IF(I.EQ.N) GO TO 190
      GO TO 180
190   X(1)=0.0
      DO 35 I=1,N  ,10
      X(I)=Y1(I)
      J=K+1
      Y(1,J,3)=C1
      Y(1,J,4)=C2
      PRINT26,X(I),Y(I,J,1),Y(I,J,2),Y(I,J,3)    ,Y(I,J,4)
26    FORMAT(2X,5F15.8)
35    CONTINUE
C
C     *************************************************************
C     ENERGY EQUATION
C     *************************************************************
C
      PR=0.0
      DO 701 II=1,2
      CALL FLUN (20000)
      PR=PP+0.5
      PRINT 111,PR
111   FORMAT (/20X,F15.8)
      DO 601 I=1,N
      P(I,1,1)=0.0
601   P(I,1,2)=0.0
171   K=0
      K=K+1
      P1(1)=0.0
      P2(1)=0.0
      P3(1)=1.0
      DO 101 I=2,N
      IF(I.GT.11) GO TO 881
      H=0.01
      GO TO 991
881   H=0.1
991   CALL RUNG(P1(I-1),P2(I-1),P3(I-1),0.0,P(I,K,2),PR,Y(I,J,1),
     1   ZZ1,X,I,K+1,A,H )
      P1(I)=X(I)
      P2(I)=ZZ1(I,K+1,1)
101   P3(I)=ZZ1(I,K+1,2)
```

```
      P3(1)=0.0
      DO 201 I=2,N
      IF(I.GT.11) GO TO 888
      H=0.01
      GO TO 999
888   H=0.1
999   CALL RUNG(P1(I-1),P2(I-1),P3(I-1),1.0,P(I,K,2),PR,Y(I,J,1),
     1   ZZ2,X,I,K+1,A,H )
      P1(I)=X(I)
      P2(I)=ZZ2(I,K+1,1)
2(1   P3(I)=ZZ2(I,K+1,2)
      P(1,K,1)=0.0
      C=(1.0-ZZ2(N,K+1,1))/ZZ1(N,K+1,1)
      DO 301 I=2,N
      P(I,K+1,1)=C*ZZ1(I,K+1,1)+ZZ2(I,K+1,1)
301   P(I,K+1,2)=C*ZZ1(I,K+1,2)+ZZ2(I,K+1,2)
      P(1,K+1,1)=0.0
      P(1,K+1,2)=C
      I=2
181   IF(ABS(P(I,K+1,1)-P(I,K,1)).LT.DELTA) GO TO 133
      DO 78 I=1,N
      P(I,K,1)=P(I,K+1,1)
      P(I,K,2)=P(I,K+1,2)
78    CONTINUE
      GO TO  171
133   I=I+1
      IF(I.EQ.N) GO TO 191
      GO TO 181
191   DO 38 I=1,N,10
      JJ=K+1
      PRINT 27,X(I),P(I,JJ,1),P(I,JJ,2)
27    FORMAT(2X,15F8.4)
38    CONTINUE
701   CONTINUE
36    CONTINUE
      STOP
      END
```

```
C     RUNGE-KUTTA INTEGRATION SUBROUTINE FOR VELOCITY PROFILE
C
$IBFTC
      SUBROUTINE RUNGA(R1,R2,R3,R4,R5,DEL,SF,S1,S2,S3,Z,X,N,V,I1,H   )
      DIMENSION  A(4),B(4),C(4),Y(6)      ,Q(10),DY(6),Z(91, 5,4),X(91
      A(1)=0.5
      A(2)=1.-(0.5)**.5
      A(3)=1.+(0.5)**.5
      A(4)=1.0/6.0
      B(1)=2.0
      B(2)=1.0
      B(3)=1.0
      B(4)=2.0
      C(1)=0.5
      C(2)=1.-SQRT(.5)
      C(3)=1.+SQRT(.5)
      C(4)=0.5
      Y(1)=R1
      Y(2)=R2
      Y(3)=R3
      Y(4)=R4
      Y(5)=R5
      DO 25 I=1,5
      Q(I)=0.0
25    CONTINUE
50    DO 10 J=1,4
      DY(1)=1.0
      S=1.0+V*(Y(1)+H/4.0)
      DY(2)=Y(3)
      DY(3)=Y(4)
      DY(4)=Y(5)
      DY(5)=-((2.*V*S3)/S+V**3*S1/S**3-V*V*S2/S**2+ S1*S2/(2.*S)+SF*
     12.*S)+V*SF*S2/(2.*S**2)+V*S1*S1/(2.*S**3)-V*V*S1*(SF-S1*(Y(1)+
     2( 2.*S**3))*DEL-(Y(2)-SF*DEL)*(S3/(2.*S)+V*S2/(2.*S**2)-V*V* S
     3  .*S**3))  -(Y(3)-S1*DEL)*(V**3/S**3+S2/(2.*S)+ V*S1/(S)**3
     4  -(SF*V**2-2.*V*V*S1*(Y(1)+H))/(2.*S**3))
     5  -(Y(4)-S2*DEL)*(-V*V/S**2+S1/(2.*S)+V*SF/(2.*S**2))
     6  -(Y(5)-S3*DEL)*(2.*V/S+SF/(2.*S))
      DO 20 I=1,5
      TEMP=A(J)*(DY(I)-B(J)*Q(I))
      Y(I)=Y(I)+H*TEMP
      Q(I)=Q(I)+3.0*TEMP-C(J)*DY(I)
20    CONTINUE
10    CONTINUE
      X(N)=Y(1)
      Z(N,I1,1)=Y(2)
      Z(N,I1,2)=       Y(3)
      Z(N,I1,3)=Y(4)
      Z(N,I1,4)=Y(5)
      RETURN
      END
```

```
C      RUNGE-KUTTA INTEGRATION SUBROUTINE FOR TEMPERATURE PROFILE
C
$IBFTC  ENG
       SUBROUTINE RUNG(R1,R2,R3,DEL,YY,PR,YP,Z,X,N,I1,V,H)
       DIMENSION  A(4),B(4),C(4),Y(6)      ,Q(10),DY(6),Z(91, 5,4),X(
       A(1)=0.5
       A(2)=1.-(0.5)**.5
       A(3)=1.+(0.5)**.5
       A(4)=1.0/6.0
       B(1)=2.0
       B(2)=1.0
       B(3)=1.0
       B(4)=2.0
       C(1)=0.5
       C(2)=1.-SQRT(.5)
       C(3)=1.+SQRT(.5)
       C(4)=0.5
C      H=0.1
       Y(1)=R1
       Y(2)=R2
       Y(3)=R3
       DO 25 I=1,3
25     Q(I)=0.0
50     DO 10 J=1,4
       S=1.0+V*(Y(1)+H/4.0)
       DY(1)=1.0
       DY(2)=Y(3)
       DY(3)=-(PR*YP*YY/(2.0*S))*DEL
      1   -(Y(3)-YY*DEL)*PR*YP/(2.0*S)
       DO 20 I=1,3
       TEMP=A(J)*(DY(I)-B(J)*Q(I))
       Y(I)=Y(I)+H*TEMP
       Q(I)=Q(I)+3.0*TEMP-C(J)*DY(I)
20     CONTINUE
10     CONTINUE
       X(N)=Y(1)
       Z(N,I1,1)=Y(2)
       Z(N,I1,2)=Y(3)
       RETURN
       END
$ENTRY
```

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